# Mechanics Lecture Notes

# 1 Lectures 4 and 5: Kinematics of a particle

## 1.1 Introduction

*Kinematics*[1] is the study of particle motion without reference to mass or force. In some ways, studying kinematics is rather artificial: in almost all realistic situations, the motion would have been produced by forces and the problem can only be solved by investigating the equations of motion appropriate to the forces acting. The study of motion produced by forces is called *Dynamics*[2]. Note that we deal with particles, which, by definition, are point-like; they can have mass (though that is not needed in kinematics) but they have no internal structure, so they cannot, for example, spin.[3]

The example of projectile flight is important, historically and in terms of applications. From our point of view, it is a first stab at tackling equations of motion, which is fundamental to all theoretical physics courses.

## 1.2 Key concepts

- Differentiation of a vector is the same as differentiation of its Cartesian components.
- Definitions of speed, velocity and acceleration in one and in three dimensions.
- Formulae for particles moving with constant acceleration in one dimension.
- Motion of projectiles.

## 1.3 Notation

*Motion on a line*:

In one (spatial) dimension, the variables are time, position or distance or displacement from a fixed point, speed or velocity, and acceleration. We make a distinction between speed and velocity even in one dimension: velocity may be positive or negative, corresponding to the particle moving (say) to the right or left; speed is the magnitude of velocity and is therefore always positive or zero. Acceleration can also be either positive or negative.[4] We denote time by $t$, position by $x$, velocity by $u$ or $v$ and acceleration by $a$. Sometimes, displacement from the original position of the particle is denoted by $s$. We might write, for example, $x(t)$ to emphasise that $x$ is a function of time.

Velocity, by definition, is rate of change of position, so

$$v = \frac{dx}{dt} \equiv \dot{x}\,.$$

The overdot always denotes differentiation with respect to time.

Acceleration, by definition, is rate of change of velocity, so

$$a = \frac{dv}{dt} \equiv \dot{v} = \frac{d^2x}{dt^2} \equiv \ddot{x}\,.$$

*Motion in space*:

In two or three dimensions, time is still $t$, and the other variables are vector quantities. We denote position by $\mathbf{r}$, or sometimes[5] $\mathbf{x}$; we might write $\mathbf{r}(t)$ to emphasise that the position is a function of time. Velocity is denoted by $\mathbf{u}$ or $\mathbf{v}$ and acceleration by $\mathbf{a}$, both vector quantities having magnitude and direction (of course).

[1] From the Greek $\kappa\iota\nu\eta\mu\alpha$, meaning motion; think of cinema. (I put this in because I thought you might like to see some Greek letters in words instead of in equations.)

[2] From the Greek $\delta\upsilon\nu\alpha\mu\iota\kappa o\varsigma$, meaning powerful.

[3] At least, not in classical mechanics; in quantum mechanics, particles can have spin — but all sorts of other strange things happen in quantum mechanics.

[4] Even in one dimension, speed and acceleration are vector quantities: they have magnitude and direction, the direction being either to the left or to the right (say).

[5] I'm going to use $\mathbf{r}$, because students say that they can't tell the difference between my handwritten $x$ and my multiplication or vector product sign $\times$.

With respect to an origin and in standard Cartesian axes, we write

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or, to save space, just $(x, y, z)$.

Velocity, by definition, is rate of change of position, so

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} \equiv \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

This last equality (equivalence), obvious though it seems, actually needs proving. Is differentiating a vector the same as differentiating its components? The answer is yes, provided that the axes are fixed (for example, provided you are not in a train accelerating or going round a bend). You can just take my word for this, or read the footnote.[6]

Speed is the magnitude of the velocity vector $|\mathbf{x}|$, which is non-negative.

As in one dimension, acceleration is rate of change of velocity, so

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}.$$

## 1.4 Constant acceleration on a line

We can obtain standard results for constant acceleration by (as is often the case) writing down the definitions and integrating the resulting differential equations. We have

$$\ddot{x} = a,$$

where $a$ is constant, so integrating once gives

$$\boxed{\dot{x} \equiv v = at + u} \tag{1}$$

where $u$ is a constant of integration corresponding to the velocity at $t = 0$. Integrating again gives

$$x = \tfrac{1}{2}at^2 + ut + x_0 \tag{2}$$

where $x_0$ is a constant of integration corresponding to the position at $t = 0$. Sometimes, this is written as

$$\boxed{s = \tfrac{1}{2}at^2 + ut}$$

where $s$ is displacement from the initial position. It is worth checking dimensions at each stage as a quality control check. The dimension of $s$ is length $(L)$, the dimension of $u$ is $L/T$ and the dimension of $a$ is $L/T^2$; substituting these into this last equation reveals it is dimensionally consistent.

Equation (2) gives distance as a function of time. We can find distance as a function of velocity by using (1) to eliminate time from (2):

$$t = \frac{v-u}{a} \Longrightarrow s = \tfrac{1}{2}a\left(\frac{v-u}{a}\right)^2 + u\left(\frac{v-u}{a}\right)$$

which simplifies to

$$\boxed{2as = v^2 - u^2} \tag{3}$$

[6]We can't make progress with this without knowing what differentiation of a vector means. Taking the usual definition in terms of limits, we have

$$\frac{d\mathbf{r}}{dt} \equiv \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h\to 0} \frac{\begin{pmatrix} x(t+h) \\ y(t+h) \\ z(t+h) \end{pmatrix} - \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}}{h} = \lim_{h\to 0} \frac{\begin{pmatrix} x(t+h) - x(t) \\ y(t+h) - y(t) \\ z(t+h) - z(t) \end{pmatrix}}{h} = \lim_{h\to 0} \begin{pmatrix} \dfrac{x(t+h) - x(t)}{h} \\ \dfrac{y(t+h) - y(t)}{h} \\ \dfrac{z(t+h) - z(t)}{h} \end{pmatrix} \equiv \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

We will see later that this formula relates the change in kinetic energy to the work done by the accelerating force.

This formula could have been obtained directly from the equations of motion by means of the following very important idea. Essentially, it is a method of changing variable in the differential equation itself using the chain rule rather than changing variable in the solution.[7]

$$\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

so

$$\frac{dv}{dt} = a \Longrightarrow v\frac{dv}{dx} = a \Longrightarrow \tfrac{1}{2}(v^2 - u^2) = a(x - x_0) \equiv as$$

The three formulae in boxes provide everything you need for constant acceleration problems.[8]

## 1.5 Constant acceleration in three dimensions

We can integrate the vector equations for constant acceleration more or less as we did in the one dimensional case. We have

$$\ddot{\mathbf{x}} = \mathbf{a}$$

so

$$\dot{\mathbf{x}} = \mathbf{a}t + \mathbf{u}$$

where $\mathbf{u}$ is the velocity at $t = 0$. Integrating again gives

$$\mathbf{x} = \tfrac{1}{2}\mathbf{a}t^2 + \mathbf{u}t + \mathbf{x}_0 \tag{4}$$

where $\mathbf{x}_0$ is the position at $t = 0$. There is no easy formula corresponding to (3).

## 1.6 Projectiles

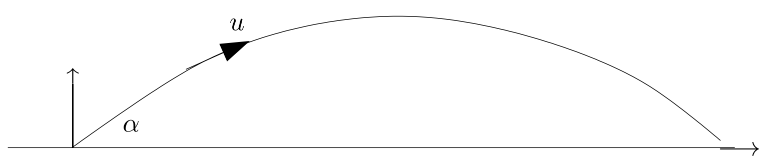
Projectiles are normally particles fired in the Earth's gravitational field. Properly, this should be treated as a problem in dynamics, since it involves forces but, since the gravitational field may be treated as uniform, the problem reduces to one of constant acceleration and the mass of the particle does not matter.[9]

*Example*

*A particle is projected from a point on a horizontal plane at speed $u$ and at angle of projection*[10] *$\alpha$. Find the equation of the trajectory.*

**First step: draw a diagram**

In most mechanics problems, the first step is to draw a good diagram with all the information in it. This may just be a good way of keeping the data in front of you, but it may also give some important insight into the problem.




[7]A similar idea occurs when dealing with orbits in the Dynamics and Relativity course: instead of solving for the position in terms of time, one changes variable in the differential equations from time to polar angle, giving a much simpler equation, the solution of which is the geometrical path in polar coordinates.

[8]Note that they work when the acceleration is *constant.* Many candidates for STEP learn these formulae and churn them out without asking themselves whether they apply.

[9]This fact will be discussed in the Dynamics and Relativity course: it is of crucial importance both in Newtonian Dynamics and General Relativity.

[10]The angle that the initial trajectory makes with the horizontal.

**Equations of motion**

The next thing to do is to write down the equations of motion, which are in this case just the mathematical expression of the fact that the acceleration the particle is given:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$$

where here $\mathbf{a}$ is the gravitational acceleration, which is constant in magnitude and direction.[11]

**Choice of axes**

The next thing to do, as in most similar problems, is to choose suitable axes and coordinates. We choose Cartesian axes with origin at the point of projection. We are free to choose the orientation of the axes, so let the $z$ axis be vertically upwards and let the $x$-axis be aligned with the initial velocity.

With this choice, the initial velocity is $(u\cos\alpha, 0, u\sin\alpha)$ and the acceleration $\mathbf{a}$ (due to gravity) is $(0, 0, -g)$, where $g$ is a positive constant.[12]

**Set out the problem in the chosen axes**

We now write down the equations of of motion in our chosen axes:

$$\mathbf{a} \equiv (\ddot{x}, \ddot{y}, \ddot{z}) = (0, 0, -g).$$

This gives us three differential equations:

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \ddot{z} = -g.$$

**Solve the equations with the given initial conditions**

Integrating each one twice and using the initial conditions $x = y = z = 0$ and $\dot{x} = u\cos\alpha$, $\dot{y} = 0$, $\dot{z} = u\cos\alpha$ at $t = 0$ gives

$$x = ut\cos\alpha, \quad y = 0, \quad z = -\tfrac{1}{2}gt^2 + ut\sin\alpha. \tag{5}$$

It can be seen, by eliminating $t$, that this is a parabola.

$$t = \frac{x}{u\cos\alpha} \Longrightarrow z = -\frac{g}{2u^2\cos^2\alpha}x^2 + x\tan\alpha. \tag{6}$$

Equation (5) is the dynamical (or kinematic) equation of the trajectory, a parabola parametrised by time $t$. Equation (6) is the geometric equation: it describes a geometric object with no sense of motion.

We could, of course, have just substituted the initial conditions $\mathbf{u} = (u\cos\alpha, 0, u\sin\alpha)$ and $\mathbf{x}_0 = 0$ into the formula (4). It is probably better, though, to start from the equations of motion rather than to quote elaborate formulae.

[11] There is of course a modelling assumption here: the Earth's gravitational field is not constant in either magnitude (it falls off as inverse distance squared) or direction (it is radial), but for the sort of projectile we normally mean, these effects are negligible.

[12] $g$ is the acceleration due to gravity at the surface of the Earth, and is about 9.8 $\mathrm{ms}^{-2}$.