Mechanics Lecture Notes

1 Lecture 7: Energy

1.1 Introduction

This lecture covers several absolutely fundamental concepts: kinetic energy, potential energy, conservation of energy, use of the equation of conservation of energy to replace one of the equations of motion. The definition of kinetic energy is generally unproblematic since it relates to the motion of the body. Potential energy is more difficult and we discuss it here only in the context of a uniform gravitational field. Similarly, the discussion of conservation of energy, which involves potential energy, is restricted to this simple case. It is important to recognise conservation of energy both as a fundamental principle and as a practical tool for investigating the motion of a body: the equation of conservation of energy is a first order differential equation (with position as the dependent variable and time as the independent variable) and can be used to replace one of the second order differential equations of motion.

1.2 Key concepts

- The work done by a force.
- Power as the instantaneous rate of doing work.
- Kinetic energy.
- Potential energy due to gravity.
- Conservation of energy as first integral of equations of motion for gravity.

1.3 Work done by a force

The work done (WD) by a force is, loosely speaking, the product of the force and the distance moved by a body as a result of the force. It represents the effort expended by the force in pushing the body along.

For a constant force in one dimension, we have simply

$$\mathrm{WD} = F(x_2 - x_1)$$

where $x_2 - x_1$ is the distance moved. If the force is a function of position, we have to integrate:

$$\mathrm{WD} = \int_{x_1}^{x_2} F(x)dx.$$

In three dimensions, when the path of the particle is a straight line from $\mathbf{x_1}$ to $\mathbf{x_2}$ and the force is constant,[1]

$$\mathrm{WD} = \mathbf{F}.(\mathbf{x}_2 - \mathbf{x}_1).$$

1.4 Power

Power, P, is the (instantaneous) rate of doing work. For a body moving in one dimension, we can express this in terms of the work done by the force in moving the body an infinitesimal distance and the time taken to do so:

$$P = \frac{\text{WD in time } dt}{dt} = \frac{F\,dx}{dt} = Fv \tag{1}$$

and the three-dimensional equivalent is $\mathbf{F}.\mathbf{v}$.

[1] We can extend the definition to varying forces on curved paths using the concept of a *line integral* which comes up in the Vector Calculus course:

$$\mathrm{WD} = \int \mathbf{F}(\mathbf{x}).d\mathbf{x}\,.$$

In general, the WD will depend on the path, i.e. on the particular curve between the two fixed endpoints traversed by the particle.

Power is measured in watts (using SI units); one watt is one joule per second, which is one newton metre per second, which is one kilogram metre^2 per second^3. For example the power of a light bulb used always to be measured in watts, and this is a measure of the rate of emitting light (related to how bright the bulb is) together with the rate of emitting heat (related to how hot it is); nowadays, you normally find a measure of brightness (in lumens) on the box.[2] so, for example, the brightness of a light bulb is a measure of its power. The power of a car, normally given as the time take to accelerate from 0 to 60 miles an hour, could also be measured in watts as in the following example. The practical 0 to 60 definition has the advantage of including information about the smoothness of the bearings and the aerodynamicness of the shape.

1.5 Worked example

A car of mass m has an engine that can work at a maximum power P. The total resistance force (including friction and air resistance) on the car at speed v is Rv.[3] *Find the time, T, it takes to accelerate from 0 to u at maximum power.*

As always, we write down the equation of motion:

$$m\frac{dv}{dt} = F - Rv = \frac{P}{v} - Rv$$

using the definition (1) of power, so

$$\int_0^u \frac{mv\,dv}{P - Rv^2} = T \Longrightarrow 2RT = m\log(1 - Ru^2/P)\,.$$

1.6 Energy

Energy comes in many different forms. We consider here just potential energy, related to the position of a particle in a force field,[4] and kinetic energy, related to the motion of a particle.[5]

We define the kinetic energy (KE) of a particle of mass m moving with velocity $\mathbf{v}$ by

$$\text{KE} = \tfrac{1}{2}m\mathbf{v}.\mathbf{v}.$$

Kinetic energy is additive, so that the KE of two particles is just the sum of the KEs of the individual particles.[6]

The potential energy (PE) of a particle in a force field relates to the ability of the particle to do work by virtue of its position in the force field. For example, water at the top of a waterfall has PE because it could be used to generate hydro-electric power. As will be discussed in the course Dynamics and Relativity, not all force fields have potential energy associated with them[7]. To avoid any complications, we will consider only the force of gravity (in fact, just a uniform gravitational field, so that gravity acts in a fixed direction, rather than, say, the gravitational field of a star which is radial), and define the PE of a particle of mass m by

$$\text{PE} = mgh, \tag{2}$$

where h is the height of the particle above some fixed level. The PE is only defined up to an additive constant, because if the fixed level is chosen a distance h_0 lower the PE will increase by mgh_0.

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[2] A more old-fashioned unit of power is the *horsepower*. There are various different definitions. The unit was invented by James Watt in order to compare the output of his steam engines with other steam engines and with that of a horse. One horsepower is about 745 watts, which was calculated by James Watt the power of a typical brewery horse. It seems surprising to me that a horse working at full power could light only 7 conventional light bulbs, but maybe that just shows how wasteful non-energy-saving bulbs are. A healthy human can produce 1.2 horsepower for a short burst and can sustain 0.1 horsepower indefinitely.

[3] Which is not very realistic: more likely, the resistive force would be modelled by a polynomial of degree 2, to take into account inertial and viscous drag and a constant friction term; but the resulting cubic denominator in the integral would not be terribly attractive for a simple example.

[4] A *force field* is a force that acts at each point in space, like gravity, but not like friction which only acts at a specific point moving with the particle. It is represented by a vector field, $\mathbf{F}(\mathbf{x})$ say, which is defined at each point $\mathbf{x}$ and defines at that point the force that would be experienced by a susceptible particle.

[5] Note, though, that what seems a completely disparate form of energy may be expressible in terms of these: for example, heat energy may be due entirely to motion of molecules which could then be regarded as kinetic energy.

[6] I expect you are wondering why KE is additive. The answer is that with this assumption the total energy of two free particles is conserved.

[7] Forces for which the concept of potential energy makes sense are called *conservative forces*; for a conservative force, a particle at a point $\mathbf{x}$ in the field has a potential energy which is uniquely determined (up to an additive constant) by $\mathbf{x}$. The defining characteristic of a conservative force is that the work done to move a particle between two fixed points is independent of the path taken.

Since the work done against gravity in raising a particle a distance d vertically upwards is mgd (force times distance moved by particle), we see that the change in PE in falling a distance h is equal to the work done in restoring it to its original position, and this coincides with the idea of PE as representing the ability to do work.

1.7 Conservation of energy

The principle of conservation of total energy, which says that energy in a closed system can be neither created not destroyed, is absolutely fundamental in classical mechanics.[8] We consider here only situations in which there are no *dissipative* forces, such as friction.[9]

It is very important to understand that in systems such as we are considering that are governed by equations of motion, conservation of energy is not an additional equation: it must be consistent with — indeed, derived from — the equations of motion. There is therefore the option of replacing one of the equations of motion with the equation of conservation of energy, and this is often a very sensible plan, for two reasons:

1. conservation of energy involves first derivatives of position (through the velocity $\frac{dx}{dt}$ in the KE) whereas the equations of motion are second order (through the acceleration $\frac{d^2x}{dt^2}$ in N2);
2. forces that do no work, such as normal reaction or tension in an inelastic string (essentially forces of constraint) appear in the equations of motion but not in the equation of conservation of energy).

In the case of a particle moving vertically in a gravitational field we can easily demonstrate conservation of energy. Conservation means that a quantity is unchanged in the motion, which here means unchanged in time. Thus to prove that total energy is conserved, we just write down the expression for total energy, differentiate it and apply the equations of motion:

$$\text{Total energy} = \text{KE} + \text{PE} = \tfrac{1}{2}mv^2 + mgz$$

where z is the height of the particle and $v = \dot{z}$. Thus

$$\frac{d}{dt}(\text{Total energy}) = mv\dot{v} + mgv.$$

But the equation of motion (N2) is $ma = -mg$, where $a = \dot{v}$, so the result follows immediately.

More generally, allowing the particle to move horizontally as well as verically, we have

$$\text{Total energy} = \text{KE} + \text{PE} = \tfrac{1}{2}m\mathbf{v}.\mathbf{v} + mgz.$$

Differentiating as before gives

$$\frac{d}{dt}(\text{Total energy}) = m\mathbf{v}.\dot{\mathbf{v}} + mg\dot{z}.$$

But now Newton's second law is $m\dot{\mathbf{v}} = (0, 0, -g)$ and, since $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$, the above expression sums to zero as before.

We have therefore just proved the fundamental result that for a single particle in a uniform gravitational field (subject to no other forces) the total energy is conserved. Conservation of energy for systems of particles (such as a solid body) and more general forces will be discussed in the Dynamics and Relativity course.

1.8 Example: particle projected vertically upwards

Suppose the particle has mass m and is projected upwards with initial speed u from position $z = 0$. Its total energy, E, is initially given by

$$E = \tfrac{1}{2}mu^2 + 0$$

[8] In elementary quantum mechanics, the governing equation describing the behaviour of a single particle (the Schrödinger equation, is essentially that of conservation of energy; in thermodynamics, the first law is precisely an expression of conservation of energy, including heat energy; in Special Relativity, conservation of energy holds but mass has to regarded as a form of energy via the famous equation $E = mc^2$; but in General Relativity, the concept of total energy is problematic, since there may not be any conserved quantities.

[9] The principle still holds even when there are dissipative forces, but then you have to take into account the mechanical energy that is converted into heat energy. Heat energy can be regarded as the kinetic energy of moving molecules but is more conveniently treated through the theory of thermodynamics .

taking the potential energy to be mgz.

Let the speed of the particle (still moving up or now falling back down — it doesn't matter which) at height z be v. Then its total energy at this point is

$$\tfrac{1}{2}mv^2 + mgz.$$

Assuming conservation of energy, we have

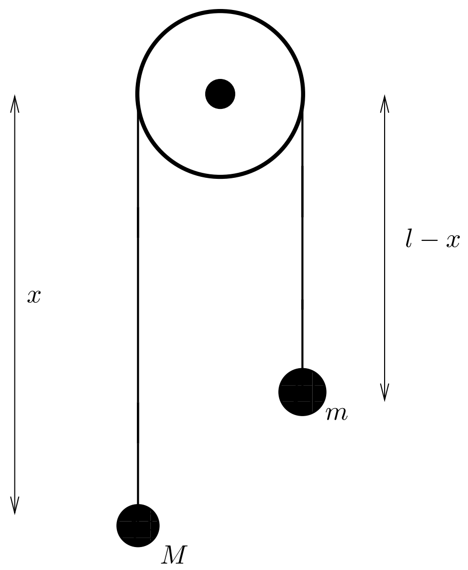
$$\tfrac{1}{2}mv^2 + mgz = \tfrac{1}{2}mu^2 \qquad \text{i.e.} \qquad v^2 - u^2 = -2gz\,.$$

which is the same kinematic equation as was derived in lecture 6. (The minus sign arise because the force in the direction of increasing z is $-g$.)

1.9 Example: the pulley from lecture 6

In this example, two particles of mass m and M are joined by a light string which passes over a smooth fixed pulley. We wish to find the acceleration of the particles.

Let x and $l - x$ be the distances of the two particles below the axle of the pulley.



We use the conservation of total energy of the two particles. The velocities of the particles (differentiating distance below the axle) are $\dot{x}$ and $-\dot{x}$ so the total kinetic energy is

$$\tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\dot{x}^2.$$

The potential energies of the particles are $-Mgx$ and $-mg(l - x)$ (the minus signs because the distances are measured downwards). Thus the total energy, which is constant, is

$$\tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\dot{x}^2 + gx(m - M) - mgl.$$

We could integrate this equation to find x as a function of t. Instead, since we want to find the acceleration, we differentiate:

$$M\dot{x}\ddot{x} + m\dot{x}\ddot{x} + g\dot{x}(m - M) = 0\,.$$

Cancelling the factor of $\dot{x}$ gives the result obtained in lecture 6 by equating forces:

$$\ddot{x} = \frac{(M - m)g}{m + M}.$$

Notice that we did not have to worry about the tension in the string at all by this method.