Mechanics Lecture Notes

1 Lecture 8: Momentum and Impulse

1.1 Introduction

This lecture covers another fundamental concept: momentum. The momentum of a body is the quantity that changes directly as a result of applying a force: the greater the force, the faster the momentum changes. This is a consequence — in fact, a statement of — Newton's second law of motion. The reason that momentum is so important is that in closed systems[1] not subject to external forces, the total momentum is conserved. This result follows from Newton's equations and will be proved in the Dynamics and Relativity course. Momentum is conserved in quite surprising situations, even when kinetic energy is not conserved, such as during explosions.

As with conservation of energy, the equations of conservation of momentum can be used very conveniently as a substitute for one or more of the equations of motion.

When one particle collides with another, they each experience a force of large magnitude for a very short duration. This is normally idealised as an infinite force for an infinitesimal time[2]. There are various ways of describing this idealisation mathematically[3] but for our purposes it is easier to deal not with the (infinite) force with a quantity called *impulse* which is, roughly speaking, the product of the very large force and the very small time during which it acts.

1.2 Key concepts

- Momentum.
- Conservation of momentum in closed systems with no external forces.
- Impulse.
- Relation between impulse and change of momentum during particle collisions.
- Newton's experimental law and coefficient of restitution.

1.3 Momentum

The *momentum* of a particle of mass m moving with velocity $\mathbf{v}$ is defined to be $m\mathbf{v}$. It is, of course, a vector quantity. The total momentum of a system of particles is just the sum of the individual momenta.

For a particle with momentum $\mathbf{p}$, Newton's second law gives

$$\frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} = \mathbf{F}$$

so if there is no force on the particle, $\mathbf{p}$ is constant: momentum is conserved. Similarly, as will be proved in the Dynamics and Relativity course, the momentum of a system of particles on which no external forces act is conserved.[4]

Thus for two colliding particles, in obvious notation (with initial velocities denoted by $\mathbf{u}$ and final velocities denoted by $\mathbf{v}$), conservation of momentum requires

$$m_1\mathbf{u}_1 + m_2\mathbf{u_2} = m_1\mathbf{v}_1 + m_2\mathbf{v_2}\,. \tag{1}$$

[1] No particles join or leave the system.

[2] In a less idealised situation, if (say) two snooker balls collide, there would be a small compression of each ball at the contact point, during which the balls instantaneously move together; then the elastic forces in the ball cause the compression to spring back to the original shape and the balls rebound. This process would take a short time and could be analysed using Newton's laws if the internal structure of the billiard balls were properly understood.

[3] For example, by means of the Dirac delta function, which you will come across in the Differential Equations course and later in the Part IB Methods course; and more rigorously later still if you study the theory of distributions.

[4] This is not quite 'similarly': it requires the use of Newton's 3rd law to show that the effects of the *internal* forces cancel out.

1.4 Newton's experimental law: coefficient of restitution

Assuming that the initial velocities are given, equation (1) gives three equations (because it is a vector equation) for six unknowns (the three components of each of the two final velocities): not enough equations!

In general the kinetic energy of the colliding bodies (for example, snooker balls) will not be conserved in a realistic collision: the fact that a collision can be *heard* tells us this immediately (some kinetic energy is converted into sound energy); and generally the internal effects will heat up the balls slightly (some kinetic energy is converted into heat energy). Therefore, conservation of kinetic energy cannot be used as an additional equation to determine the outcome of a collision.

However, we have one more tool in the bag. *Newton's experimental law* furnishes us with one further equation, which is sufficient in the case of head-on collisions in one dimension or in the case of a particle bouncing on a plane, which are the only two situations we will cover here. By experiment, Newton discovered that

$$\text{relative velocity after collision} = -e \times \text{relative velocity before collision} \qquad (2)$$

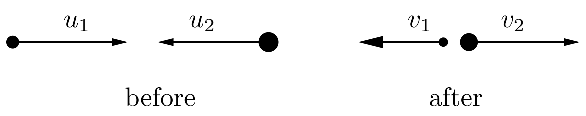
where e is a constant, called the *coefficient of restitution*, that depends on the two colliding particles (or bodies) and not, for example, on their velocities. The term *relative velocity* just means the velocity of one particle with respect to the other; i.e. in moving axes chosen so that the velocity of the second particle is zero.

If the two bodies coalesce, like putty, the relative velocity after the collision is zero, which corresponds to $e = 0$; this is called an *inelastic* collision. If the relative speed before is equal in magnitude to the relative speed after, which corresponds to $e = 1$, the collision is called *perfectly elastic*. In this case, no energy is lost in the collision: the total kinetic energy is the same as before (and you can't hear the collision!).

One has to be quite careful with the signs when applying (2), but normally common sense prevails.

1.5 Worked examples

(i) *Two particles of masses m_1 and m_2 collide head-on. Their initial velocities are u_1 and $-u_2$ and their final velocities are $-v_1$ and v_2. The coefficient of restitution between the two particles is e. Find the final speeds in terms of the initial speeds, the masses and e.*



In the picture, we are assuming that that u_1 and u_2 are positive — but it doesn't matter provided the velocities are such that the particles do in fact collide. We are also assuming, in the picture, that v_1 and v_2 are positive though they may not be: if m_2 is much larger than m_1, one could easily imagine both particles moving to the left after the collision.

Conservation of momentum in the positive x direction (to the right in the picture) gives

$$m_1u_1 - m_2u_2 = -m_1v_1 + m_2v_2.$$

Newton's experimental law, taking the velocity of the second particle relative to the first and again taking positive velocity to mean motion to the right, gives

$$v_2 - (-v_1) = -e[-u_2 - u_1]$$

Solving these two equations simultaneously gives

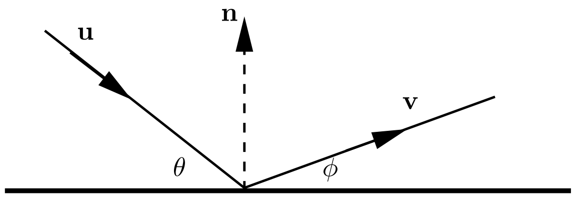
$$v_1 = \frac{(em_2 - m_1)u_1}{m_1 + m_2} + \frac{(e+1)m_2u_2}{m_1 + m_2} \qquad \text{and} \qquad v_2 = \text{Something similar - it really doesn't matter.}$$

Note that in the case $e = 1$ a direct calculation would show[5] that $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$; i.e. kinetic energy is conserved, as claimed above.

[5] I didn't do it, but I know it must: please don't spend your time on this unrewarding calculation.

(ii) *A particle is projected at velocity* $\mathbf{u}$ *on a horizontal smooth table. It hits a smooth vertical barrier, its trajectory making an angle of* θ *with the barrier. It rebounds with velocity* $\mathbf{v}$ *at an angle* ϕ *to the barrier. The coefficient of restitution between the particle and the barrier is* e*. Find* $\mathbf{v}$ *in terms of* $\mathbf{u}$ *and* e*.*

The first step, as so often in dynamics, is to choose sensible axes.[6] Let us take $\mathbf{u} = (u\cos\theta, u\sin\theta)$ and $\mathbf{v} = (v\cos\phi, v\sin\phi)$ which corresponds to taking the unit normal to the barrier to be $(0, 1)$, as shown in the diagram (which is a bird's eye view of the table).



Now we consider components of velocity and momentum parallel and perpendicular to the barrier. The important thing to realise is that parallel to the barrier *nothing happens*: it is as if the barrier were not there. The barrier exerts no force parallel to itself, so momentum is conserved.

$$u\cos\theta = v\cos\phi.$$

The perpendicular components of velocity satisfy Newton's experimental law:

$$v\sin\phi = eu\sin\theta$$

and these two equations are sufficient to determine ϕ and v in terms of θ and u.

1.6 Impulse

In the example above of a bouncing particle, the momentum of the particle is not conserved: indeed, the normal component changes direction as well as magnitude. Why not?

The answer can only be that external forces act on the system and a moment's thought reveals that this is the case: there must be a force holding the barrier in place on the table. The force acts for a very short time and must be very large to turn the particle round.

In this situation, and other similar situations which involve very large forces acting for very short time, it is convenient work with the time-integrated force, which is called the *impulse*:

$$\int \mathbf{F}dt \equiv \mathbf{I}.$$

The impulse can be thought of as a measure of the amount by which momentum fails to be conserved. Using Newton's second law, we have:

$$\mathbf{I} = \int m\frac{d\mathbf{v}}{dt}dt = \Delta(m\mathbf{v})$$

i.e. impulse equals change of momentum. Thus in the above example, the barrier feels an impulse equal to $m(v\sin\phi + u\sin\theta)$, where m is the mass of the particle, in the direction normal to the barrier, and the particle feels an impulse equal in magnitude but in the opposite direction. There must be some external impulse of magnitude $m(v\sin\phi + u\sin\theta)$ acting on the system to prevent the barrier from moving.

[6] We could, if we were feeling masochistic, stick with vectors (no components), giving the normal to the barrier the direction $\mathbf{n}$, and $u\sin\theta = \mathbf{u}.\mathbf{n}$, etc. However, we can easily go back into vectors after we have solved the problem.