Mechanics Lecture Notes

1 Lecture 9: Simple harmonic motion

1.1 Introduction

This lecture covers simple harmonic motion (SHM). SHM occurs when a particle or body is displaced from an equilibrium position and experiences a restoring force (i.e. a force tending to pull it towards the equilibrium point) that is proportional to the displacement from the equilibrium point. It may seem that this would only occur in rather special conditions, but in fact it is ubiquitous: at the equilibrium point, the force acting on the particle is zero (by definition), so for a *small* displacement, the force is approximately linear (assuming that the force can be expanded in a Maclaurin series about the equilibrium point).

The second order differential equation that arises from the linear force law is easily solved, giving sin/cos or complex exponential solutions, and these describe the oscillatory motion with period independent of amplitude characteristic of SHM.

Two important examples are discussed: a particle hanging on a spring, which is the model for any one-dimensional SHM situation; and the simple pendulum. which will arise in the Dynamics and Relativity course, in the rotating frame of the Earth, in the form of the Foucault pendulum.

1.2 Key concepts

- Simple harmonic motion as a consequence of a linear restoring force: period and frequency.
- Hooke's law, which implies a linear restoring force when elastic materials are deformed.
- Derivation of equations of motion for the spring and simple pendulum.

1.3 Simple harmonic motion

Simple harmonic motion occurs when a particle experiences a force that is proportional to its displacement from a fixed point, and the constant of proportionality is negative. Choosing a sensible coordinate x to be the distance from the fixed point,[1] the equation of motion, using Newton's second law, is

$$m\frac{d^2x}{dt^2} = -kx \tag{1}$$

where $k > 0$. This is one of the few equations for which you need the solution at your finger tips. It is

$$x = A\cos\omega t + B\sin\omega t \equiv a\cos(\omega t + \epsilon) \equiv Pe^{i\omega t} + Qe^{-i\omega t}, \tag{2}$$

where $\omega = \sqrt{k/m}$. These three forms are completely equivalent, though it may be convenient in any given situation to prefer one of them. The solution is periodic and the quantity ω is called the *angular frequency*[2] of the motion. The *period* is the time taken for the cycle to repeat, namely $2\pi/\omega$; the *frequency* is $(\text{period})^{-1}$. A graph of x against t would show a wave.

In the $a\cos(\omega t + \epsilon)$ form of the solution, a is called the *amplitude*, $\omega t + \epsilon$ is called the *phase*, and ϵ may be referred to as the *phase shift*.

The SHM equation (1) has a first integral which can be obtained by writing

$$\frac{d^2x}{dt^2} \equiv \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx}$$

or, equivalently, by multiplying the equation by $\dot{x}$:

$$m\dot{x}\ddot{x} = -k\dot{x}x$$

and integrating:

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E \tag{3}$$

where E is a suggestively named constant of integration. The first term in (3) has the obvious interpretation of the kinetic energy of the particle. It would not be surprising, therefore, if the

[1] Which might require a transformation of the form $x \to x - x_0$ to shift the origin given in the problem to the equilibrium point.

[2] measured in radians per second if t is in seconds

second term could be interpreted as its potential energy, and this is easily demonstrated. Recall that potential energy is related to the ability of the particle to do work. The work done on the particle to move it from the equilibrium point to displacement x is

$$\int (-F)\, dx = \int (kx)\, dx = \tfrac{1}{2}kx^2$$

as expected.

We can check that the solution (2) satisfies (3) (it must, of course):

$$x = a\cos(\omega t + \phi) \Longrightarrow \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}ma^2\omega^2\sin^2(\omega t + \phi) + \tfrac{1}{2}ka^2\cos^2(\omega t + \phi) = \tfrac{1}{2}ka^2$$

using $k = m\omega^2$. Thus $E = \frac{1}{2}ka^2$ which is constant, as required.

1.4 Hooke's Law

Hooke wrote down his famous law, somewhat cryptically, in 1676 in the form *ceiiinosssttuv*. This turned out to be an anagram of the Latin 'ut tensio sic vis' which translates to 'as the extension, so the force'.[3] In more helpful language, this would express the idea that strain (i.e. deformation, stretching, compression, etc, divided by natural length) is proportional stress (i.e. applied force per unit area).

This experimental law holds for a wide variety of materials under a wide range of conditions; for example, it holds for steel wire until what it called the elastic limit is reached and the wire starts to stretch like plasticine. It does not hold for rubber[4].

1.5 A note on springs and strings

Springs and elastic strings are fundamentally different: the restoring force in a string is tension due to longitudinal stretching; the restoring force in a spring is due to bending and twisting of the material from which the spring is made. Furthermore, a string can go slack, which is a godsend to setters of STEP problems, whereas if a spring is compressed so that it is shorter than its natural length it exerts a restoring force towards the equilibrium. For both string (when it is not slack) and spring, to a good approximation, the restoring force is proportional to the extension.

In the case of a string, the restoring force comes from Hooke's law:

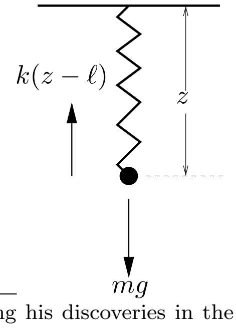
$$\text{tension} = \text{modulus of elasticity} \times \text{extension/natural length.}$$

The natural length of the string occurs in the denominator because the definition of 'strain' in Hooke's law is extension/natural length. The modulus of elasticity of the string takes into account its cross-sectional area, which is assumed to be constant. This explains why the left hand side of the equation is a force (tension), not a force per unit area (stress) as in Hooke's law.

For a spring, the restoring force is just $k \times$ extension, where k is the spring constant.

1.6 Example: particle hanging on a spring

A particle of mass m is attached to the lower end of a spring the upper end of which is fixed. The spring has unstretched length l and spring constant (i.e. the constant of proportionality between restoring force and extension of the spring) k. Initially, the spring hangs vertically at rest. The mass is then displaced a distance a downwards and released.



[3] Newton was also not averse to publishing his discoveries in the form of anagrams: the advantage was that he could claim priority in case anyone else discovered the same thing later, but for the moment it was kept secret to prevent others developing the idea.

[4] When you blow up a balloon it is hard to start then gets easier.

Let z be the distance of the mass below the top of the spring, to that the extension of the spring is $z - l$. Then the equation of motion of the particle is

$$m\ddot{z} = -k(z - l) + mg.$$

In equilibrium, $\ddot{z} = 0$, the equilibrium position is given by $z = l + mg/k$. If we choose a new coordinate x to be the downwards displacement from equilibrium, i.e. $x = z - (l + mg/k)$, we have[5]

$$m\ddot{x} = -kx$$

which is SHM. The solution is

$$x = A\cos\omega t + B\sin\omega t.$$

Initially, $x = a$ and $\dot{x} = 0$, so

$$x = a\cos\omega t \quad \text{and} \quad z = a\cos\omega t + l + mg/k.$$

[5]We don't need to do this: we could solve the differential equation directly for z, using a (constant) particular integral and a complementary function.