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String Theory

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Abstract: Lecture Notes to accompany the Part III String Theory Course, 2020. Please send any comments or corrections to the email address above.

Contents

Books and Lecture Notes

There are lots of good resources available to someone wanting to learn the basics of string theory. Below is an incomplete sampling. You should certainly not rely only on these lecture notes - books are there to be read!

Good 'serious' books are:

- String Theory Vol 1, J. Polchinski, CUP
- Superstring Theory Vol 1, M.B.Green, J.H. Schwarz & E. Witten, CUP
- Basic Concepts of String Theory, R. Blumenhagen, D. Lust & S. Theisen
- String Theory and M-Theory, K. Becker, M.Becker & J.H. Schwarz
- A Primer on String Theory, V. Schomerus, CUP

There are also many excellent sets of lecture notes available for free:

- David Tong's (excellent) Lecture notes
  http://www.damtp.cam.ac.uk/user/tong/string.html
- What is String Theory, Joe Polchinski
  https://arxiv.org/abs/hep-th/9411028
- Introduction to Superstring Theory, Elias Kiritsis
  https://arxiv.org/abs/hep-th/9709062

Many, many more can be found at: https://www.stringwiki.org

More popular books that perhaps give you some flavour of the history of the development of the theory are

- The Elegant Universe, B. Greene, Vintage
- Why String Theory?, J. Conlon, CRC Press
- Little book of String Theory, S. Gubser, Princeton University Press
- Also see this great article introducing string theory:
  https://physicstoday.scitation.org/doi/10.1063/PT.3.2980

Conventions

In these notes we shall adopt the following conventions:

- We will take the metric of spacetime to be 'mostly plus', i.e.

$$\eta_{\mu\nu} = \mathrm{diag}\{-1, +1, +1, ..., +1\}$$

- $\hbar = 1 = c$.

1 Introduction *or* Expectation Management

The need for a new theory

There are many reasons why a new type of theory, that goes beyond classical general relativity and quantum field theory is needed.

- What choses the parameters in the Standard Model?
- What choses the cosmological constant to be so small?
- The failure of naive gravitational perturbation theory at loop order.
- Classical GR breaks down at singularities.
- The black hole information paradox
- ...

1.1 Conceptual Obstacles

- The nature of time in quantum gravity
- How do you quantise without a pre-existing causal structure?
- What are the gauge-invariant observables? (There are no local diffeomorphism-invariant observables).

Let us consider the second issue. Let us consider the example of a scalar field $\phi(\mathbf{x}, t)$ with Lagrangian $\mathcal{L}$. Given a natural notion of time we identify the canonically conjugate momentum $\Pi(\mathbf{x}, t)$ and impose the canonical commutation relations are

$$[\Pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = i\delta^D(\mathbf{x} - \mathbf{y})$$

if $\mathbf{x}$ and $\mathbf{y}$ are time-like separated (i.e. if they are in causal contact). We ask that all fields commute at space-like separation.

For gravity, we might take the fundamental field to be the metric $g_{\mu\nu}(\mathbf{x}, t)$ and the action to be the Einstein-Hilbert action

$$S[g] = \frac{1}{\kappa_D} \int \mathrm{d}^D x \sqrt{-g} R$$

Given that the metric itself defines the causal structure, how are we to define the fundamental canonical commutation relations in a background-independent way?

These are weighty questions and we need clues to make progress. Given the absence of experimental data, our reliance on clues from theory are even more important than usual. We can als consider more practical problems

1.2 Technical or Practical Obstacles

One way to avoid these issues in the first instance is to take a note from interacting quantum field theory.

Choose a classical background and look at quantum perturbations of this background. The background metric defines the causal structure with which we can define a consistent quantum perturbation theory. For instance, we might look at deviations from flat spacetime and take

$$g_{\mu\nu}(\mathbf{x}, t) = \eta_{\mu\nu} + h_{\mu\nu}(\mathbf{x}, t).$$

This also gives an answer to the third question: the observable is the graviton S-matrix.

The diffeomorphism invariance acts, to first order as

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + ...$$

Since the Ricci scalar includes dependence on the inverse metric writing the Einstein-Hilbert action in terms of $h_{\mu\nu}$ involves an infinite number of terms (the action for the graviton is non-polynomial). Fixing a gauge, we find

$$S[h] = \int \mathrm{d}^D x \, \left(h^{\mu\nu} \Box h_{\mu\nu} + ... \right).$$

As with most interacting quantum field theories we proceed by using the action to determine Feynman rules which we use to calculate to a given order in perturbation theory. The quadratic term determines the propagator.

Divergences at loop order. Not renormalisable! Even this pragmatic approach seems to fail. It seems we need a different starting point.

Alternatives

Though arguably the most developed and best understood, string theory is arguably not the only game in town.

- QFT in curved spacetime

  Whilst not a proposal for a quantum theory of gravity, this does explore some of the issues outlined above, such as the black hole information paradox.

- Loop Quantum Gravity
- Causal Set Theory
- ...

What is string theory?

We don't really know.

What *do* we know?

In a minimal sense it seems to be the perturbation theory for a specific quantum theory (M-theory) which has a number of ten-dimensional classical vacua. The perturbation theory around such vacua is described by String Theory. There is evidence that the underlying theory has classical vacua of different kinds which whose perturbation theory is not described by any string theory. For example, one such vacuum is four-dimensional spacetime and the perturbation theory is governed by $\mathcal{N} = 4$ Super Yang-Mills. None of this is very helpful as I haven't told you what M-Theory is.

A more accessible, but possibly more misleading starting point is the usual one taken by popular science books: Imagine the fundamental objects of nature are tiny vibrating strings where different harmonics correspond to different fundamental particles. This includes the graviton.

But hang on, isn't a graviton 'just' a perturbation in spacetime? How can we distinguish the object from the spacetime it lives in? The split is one of convenience and is, at a fundamental level, arbitrary. In gravitational perturbation theory we make a split between the background metric (which we presumably know a lot about) and a perturbation

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x).$$

The description we have from string theory is fundamentally perturbative.

So, the vibrational modes of the string are fundamental particles in some perturbative sense. And just as particles sweep out world-lines, strings sweep out surfaces, which we shall call world-sheets. Thus, we might expect pictures like this to have something to do with Feynman rules.

What makes this start to be interesting is that there is a good understanding for how these things interact in perturbation theory. There is a set of rules that look just like Feynman rules. There are propagators and there are interaction vertices For a given 'allowed' classical solution (we shall discuss what equations this is a solution of later on), there is a set of Feynman rules that allow us to calculate scattering amplitudes of perturbations of the background. The Feynman diagrams of this mysterious theory are given by plumbing together such two-dimensional surfaces (there are strict rules of how to do this in a way that is consistent with the underlying

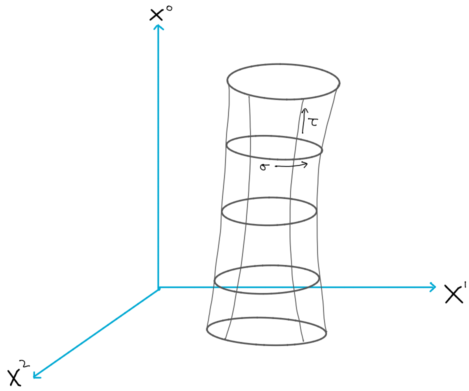


Figure 1. The embedding $X \to M$, given by $X^\mu(\sigma, \tau)$

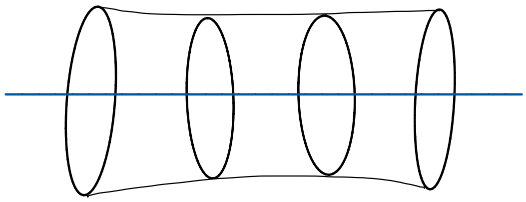


Figure 2. A closed string propagator

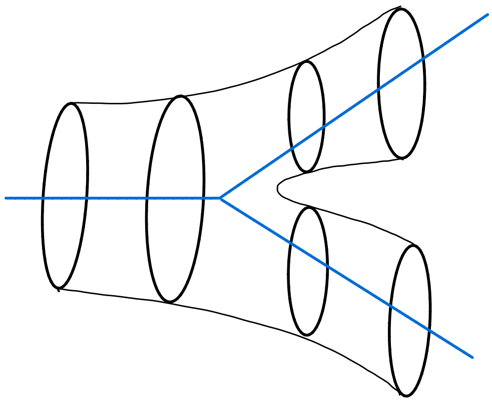


Figure 3. A three closed string vertex

symmetries of the theory). The asymptotic states are given by vibrational harmonics of the string.

So the picture we have is that given by figure 4.

Thus our starting point is to study the embedding of the two-dimensional surfaces Σ into an ambient, or *target* space M.

$$X : \Sigma \to M$$

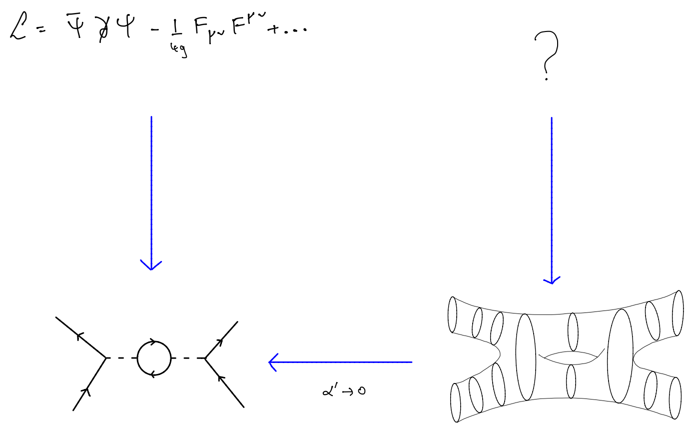


Figure 4. Where does string theory come from?

If we have coordinates X^μ is some patch of M, we make this embedding concrete by putting coordinates on the surface Σ. Let's call them $\sigma^a = (\tau, \sigma)$, so that we can describe the embedding by the set of functions

$$X^\mu = X^\mu(\tau, \sigma), \qquad \mu = 0, 1, 2, ...D - 1.$$

If we have an action functional for the embedding $S[X]$ then we can try to quantise the embedding. Things start to get special when we notice that the two-dimensional quantum field theory living on the surface of the these 'world-sheets' has a rather beautiful structure - it is a Conformal Field Theory. Moreover, there is a one-to-one correspondence between states of the CFT and operators that describe deformations in the background. These deformations include deformations of the metric - they include gravitons. As such this theory includes perturbative quantum gravity. Moreover, one can argue that there are no problematic UV divergences in this theory. There are some potential IR issues, but we will come to that later.

In units in which $c = \hbar = 1$, the Planck length is

$$\ell_p = \sqrt{\hbar G_N / c^3} = 1.6 \times 10^{-35}\text{m}.$$

The characteristic length scale for the string is usually taken to be larger than this, thus justifying using a classical spacetime as the background for the perturbation theory.

Many questions are answered (unification, how to do gravitational perturbation theory), some are strongly hinted at (there is no bh paradox - all evolution is unitary), whilst others are not engaged with (in perturbation theory we have a pre-existing causal structure to play with).

Many new questions are raised, such as the nature and significance of spacetime.

Part I

The Classical String and Canonical Quantisation

What to quantise? The standard approach to relativistic quantum theory is second quantisation and with good reason - the approach is responsible for much of our understanding of the Standard Model of particle physics and many advances in more speculative quantum field theories. However, it is not the only game in town. In this section we briefly champion the first quantised approach.

Second Quantization $(\widehat{X}^i, t) \to x^\mu = (x^i, t)$ Space and time are parameters and the physical objects are operators $\widehat{\Phi}(x^i, t)$.

First Quantization $(\widehat{X}^i, t) \to \widehat{X}^\mu = (\widehat{X}^i, \widehat{T})$ We elevate both space and time to operators and introduce parameters σ^a to describe the theory. These operators describe the embedding of surfaces (worldlines, worldsheets, worldvolumes, etc) into spacetime. We expect one of these parameters, call it τ to play the role of time on the parameter surface. The operators may then describe the embedding of the parameterised surface in spacetime. In the simplest case we may have a single parameter and the surface is a worldline with embedding

$$\Big(X^i(\tau), T(\tau)\Big).$$

Second quantisation is the route we take in conventional quantum field theory. It is successful. One can deal with the physics of vacua, such as finding low energy minima and symmetry breaking. Feynman rules can be easily derived and are not introduced in an ad-hoc way and off-shell physics can be dealt with in a natural way. However, first quantisation has had some success. It is arguably the framework that allows more rigorous calculations to be done, one can study anomalies rigorously, and there has been recent progress in the calculation of scattering amplitudes that suggest Feynman rules are not the most sensible way to calculate and great simplifications can be achieved by looking at formalisms that are closer in spirit to first quantisation.

2 Particles

We start with a more familiar example - the relativistic particle. This will serve as a toy model for the relativistic string and many of the ideas central to the classical string will be on display here, albeit in a simpler setting.

2.1 Minimising worldline distance

We shall work throughout in a 'mostly plus' metric convention in spacetime; i.e. $\eta_{\mu\nu} = \mathrm{diag}\{-1, +1, +1, ..., +1\}$. Imagine a massive particle in flat spacetime with background metric $\eta_{\mu\nu}$ travelling between two points x_1^μ and x_2^μ. It sweeps out a worldline $\mathcal{L}$, the length of the which defines an action functional

$$S[X] = -m \int_{s_1}^{s_2} \mathrm{d}s$$

where s is some parameter along $\mathcal{L}$.

We can parameterise by τ, such that $x_1 = X^\mu(\tau_1)$ and $x_2 = X^\mu(\tau_2)$ for some function $X^\mu(\tau)$. We may write this as

$$S[X] = -m \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \tag{2.1}$$

where the dot denotes a derivative wrt τ. The constant m must have dimensions of mass so we assume that this is the mass of the particle. The conjugate momentum is given by

$$P_\mu(\tau) = \frac{\partial L}{\partial \dot{X}^\mu} = -m \frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}},$$

so that the mass-shell condition

$$P^2 + m^2 = 0,$$

is satisfied identically. We see then that this formalism is manifestly on-shell.

The physics of the action above should be independent under a choice of parametrisation. We can see this if we change $\tau \to \tau + \xi(\tau)$. The embedding changes as

$$X^\mu(\tau) \to X^\mu(\tau + \xi(\tau)) = X^\mu(\tau) + \xi \dot{X}^\mu(\tau) + ...$$

and so to first order

$$\delta X^\mu(\tau) = \xi(\tau)\dot{X}^\mu(\tau).$$

The action is indeed invariant under such a transformation if the variation vanishes at the end points.

2.2 Metric formalism

What about massless particles? Simply setting $m \to 0$ in (2.1) is not very helpful. We introduce an auxiliary field, the einbein[1] $e(\tau)$ and instead consider the action

$$\boxed{S[X,e] = \frac{1}{2}\int_{\mathcal{L}} \mathrm{d}\tau \Big(e^{-1}\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu - em^2 \Big)} \tag{2.2}$$

We shall show that this action is classically equivalent to (2.1) and shall assume that it gives rise to an equivalent quantum theory. One of the many nice things about the action (2.2) is that we can sensibly talk about massless particles.

Using the Lagrangian

$$L = \frac{1}{2}\Big(e^{-1}\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu - em^2 \Big),$$

the equations of motion are given by the Euler-Lagrange equations, such as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) - \frac{\partial L}{\partial X^\mu} = 0,$$

which tells us that the X^μ equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(e^{-1}\dot{X}^\mu \Big) = 0.$$

The $e(\tau)$ equation of motion gives

$$\dot{X}^2 + e^2 m^2 = 0, \tag{2.3}$$

The key point is that the equation of motion for $e(\tau)$ is algebraic and so it is really a constraint (think Lagrange multiplier) and we can substitute it back into the action to recover the original action.

The momentum conjugate to X^μ is

$$P_\mu \equiv \frac{\partial L}{\partial \dot{X}^\mu} = e^{-1}\dot{X}_\mu,$$

and so, written in terms of the momenta, the constraint equation (2.3) becomes the mass-shell condition

$$P^2 + m^2 = 0,$$

i.e. the constraint tells us something about the *spacetime* physics.

[1]The terminology comes from General Relativity. It is often useful (especially when describing spinors) to encode the metric degrees of freedom in the field ${e_\mu}^\alpha$, where $g_{\mu\nu}(x) = \eta_{\alpha\beta}{e_\mu}^\alpha(x){e_\nu}^\beta(x)$. In four-dimensions, these *vierbein* have one leg in the tangent space and one leg on the spacetime, hence the terminology. In one-dimension the same language may be used but the construcution is somewhat redundant.

For a time-like vector in this signature, we have $\dot{X}^2 < 0$ and so

$$e^{-1} = \frac{m}{|\dot{X}|},$$

Substituting back into the action

$$\begin{aligned}
S[X, e(X)] &= \frac{1}{2}\int_{\mathcal{L}} \mathrm{d}\tau \; e^{-1}\Big(\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu - e^2 m^2\Big) \\
&= -\int \mathrm{d}\tau \; e^{-1}|\dot{X}|^2 \\
&= -m\int \mathrm{d}\tau \; |\dot{X}| \\
&= -m\int \mathrm{d}\tau \; \sqrt{-\dot{X}^2}
\end{aligned}$$

which is the previous action.

2.3 Symmetries

- Worldline reparameterization acts as

$$\delta X^\mu = \xi \dot{X}^\mu, \qquad \delta e = \frac{\mathrm{d}}{\mathrm{d}\tau}(\xi e).$$

  We use the reparameterisation invariance to fix the einbein $e(\tau)$ to be whatever we like (using the arbitrary functional dependence of $\xi(\tau)$ to remove the degrees of freedom in $e(\tau)$). This is just like using gauge invariance to remove the longitudinal polarisation of the photon.

- There is also the rigid[2] symmetry

$$X^\mu(\tau) \to \Lambda^\mu{}_\nu X^\nu(\tau) + \xi^\mu, \qquad \Lambda^\mu{}_\nu \in SO(D-1, 1),$$

  so the theory is naturally invariant under the Poincare symmetries of spacetime.

The $X^\mu(\tau)$ equation of motion is

$$\ddot{X}^\mu(\tau) = 0,$$

telling us that free particles move in straight lines in flat spacetime. It is the mass-shell constraint that is telling us whether the particle is time-like or space like, thus both the $X^\mu(\tau)$ and the $e(\tau)$ equations of motion play a crucial role in determining the physics of the theory.

[2] We will use the terms 'rigid' and 'global' symmetry interchangably. In this context our matrices $\Lambda^\mu{}_\nu$ are independent of τ.

If we were to quantise the theory by introducing equal τ commutators

$$[X^\mu(\tau), P_\nu(\tau)] = i\delta^\mu_\nu,$$

we could then construct the Hilbert space of physical states $\mathcal{H}$. The mass-shell condition, which came from the $e(\tau)$ equation of motion, must then be imposed as a constraint on the states in $\mathcal{H}$

$$(P^2 + m^2)|\Phi\rangle = 0.$$

2.4 Comments

Curved Backgrounds We could use this formalism to describe the motion of a massless particle on a curved spacetime with metric $g_{\mu\nu}(X)$

$$S[X,e] = \frac{1}{2}\int_{\mathcal{L}} \mathrm{d}\tau e^{-1} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu$$

As one might expect, extremising the action leads to the geodesic equation

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\lambda}\dot{X}^\nu \dot{X}^\lambda = 0$$

If we choose normal coordinates about a point x_0, where $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$, then the metric becomes

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + (x^\lambda - x_0^\lambda)\partial_\lambda g_{\mu\nu}(x_0) + ...$$

or even better using Riemann normal coordinates, where $x = x_0 + y$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}(x_0)y^\lambda y^\rho - \frac{1}{6}\nabla_\rho R_{\mu\lambda\nu\kappa}(x_0)y^\rho y^\lambda y^\kappa + \mathcal{O}(y^4),$$

then the action looks like that of a free theory with a number of interaction terms describing the curvature of the background and so, in principle, provides a perturbative way to study particles moving in curved spacetime.

Quantum Gravity? This describes the motion of a particle on a curved background. If we quantize we find this gives a description of a quantum particle on a classically curved background. The physics of the worldline theory does not include deformations of the background. We have identified the momentum above, so quantisation proceeds in the usual manner by imposing the canonical equal time commutation relations of quantum mechanics

$$[X^\mu(\tau), P_\nu(\tau)] = i\delta^\mu_\nu.$$

We see that a one-dimensional quantum field theory gives quantum mechanics. Suppose we want to deform our massless theory from a flat background to a curved one

$$\eta_{\mu\nu} \to g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

We could do so by adding into the action the term

$$S[X,e] \to S[X,e] + \frac{1}{2}\int_{\mathcal{L}} \mathrm{d}\tau e^{-1}\mathcal{O}(X,\dot{X})$$

where $\mathcal{O}$ is the appropriate operator. This will not describe quantum gravity in the background as the Hilbert space of the quantum mechanical theory does not contain a state in its Hilbert space corresponding to the operator

$$\mathcal{O}(X,\dot{X}) = h_{\mu\nu}(X)\dot{X}^{\mu}\dot{X}^{\nu}$$

that deforms the metric of spacetime. All we have are plane waves. Put another way, from the worldline quantum mechanics perspective this is not a physical deformation of the existing theory, the worldline in the new background is really a completely different theory altogether. We shall return to this idea later. If we followed this through, we would be doing QFT on a curved background, not quantum gravity.

3 Classical Strings

We now look at our man object of interest: the embedding of a two-dimensional surface Σ into a D-dimensional spacetime M with coordinates X^{μ}, where $\mu = 0,1,2,...,D-1$. We describe this surface by the embedding of Σ into M

$$X : \Sigma \to M.$$

With this in mind, we often refer to M as the *target space.* To make this concrete, we choose a parameterization (σ,τ) and describe the embedding by the functions $X^{\mu}(\sigma,\tau)$. What is the physics of this embedding? Classically we expect it to minimise the area (think of a soap bubble). The justification will be aposterori.

We shall only consider closed strings, i.e. those for which

$$X^{\mu}(\tau,\sigma+2\pi) = X^{\mu}(\tau,\sigma).$$

The generalisation to open strings (worldsheets with boundary) is straightforward once you have understood the closed string. The closed sector contains gravity and so will be the primary focus in this course. The attitude we shall take is that, once you have mastered the bosonic closed string, you are well on the way to understanding the open and supersymmetric strings.

3.1 The Nambu-Goto Action

Following on from the worldline action (2.1), it is natural to propose the following action for the relativistic string

$$\boxed{S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}\tau\,\mathrm{d}\sigma\sqrt{-\det(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)}} \tag{3.1}$$

α' is a constant (the only free parameter in the theory). It has dimensions of spacetime area. The string length is often introduced

$$\ell_s = 2\pi\sqrt{\alpha'}$$

and we usually justify the perturbation theory on a classical spacetime by assuming $\ell_P \ll \ell_s$. We often also speak of the string tension[3]

$$T = \frac{1}{2\pi\alpha'}$$

The action is independent of the parameterization we use. The object

$$G_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

is clearly the induced metric on the worldsheet. The square-root in this action makes it difficult to work with. A much better starting point is:

3.2 The Polyakov Action

The Polyakov action is[4]

$$\boxed{S[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\;\sqrt{-h}h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu} \tag{3.2}$$

This is a two-dimensional generalisation for the action (2.2). The equations of motion may be found in the usual way

The h_{ab} equations of motion: The response of the action to a change in the worldsheet metric is given by the stress tensor T_{ab}.

$$\delta S = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}T_{ab}\delta h^{ab},$$

The $h^{ab}(\sigma,\tau)$ equation of motion[5]

$$\frac{4\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{ab}} = T_{ab}$$

[3] Not to be confused with the stress tensor!

[4] We will later choose a Euclidian signature and so replace $\sqrt{-h}$ with $\sqrt{h}$, so more properly we should write $\sqrt{|h|}$ in place of $\sqrt{-h}$ in the above action.

[5] The $\sqrt{h}$ is not usually included in field theory but has become a standard convention in string theory.

is simply the vanishing of the stress tensor:

$$\boxed{T_{ab} = 0},$$

where the stress tensor is given by

$$\boxed{T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu\right)}.$$

We shall see that this is one of the most important equations in string theory. We notice that the trace of the stress tensor vanishes identically in two-dimensions (since $h^{ab}h_{ab} = 2$)

$$h^{ab} T_{ab} = 0.$$

We should think of the vanishing of the stress tensor as the stringy generalisation of the mass-shell condition coming from the einbein equation of motion in (2.2). In particular, we can decompose the stress tensor into harmonic modes, each of which must vanish, and the vanishing of the zero mode will be the mass-shell condition for the string. The string contains many states in its spectrum and the vanishing of the other modes will impose appropriate physical constraints (such as an absence of longitudinal polarisations in massless states).

The X^μ equations of motion: The equation of motion for the embedding fields X^μ is

$$\boxed{\frac{1}{\sqrt{-h}} \partial_a \left(\sqrt{-h} h^{ab} \partial_b X^\mu\right) = \Box X^\mu = 0}$$

3.2.1 Classical Equivalence of the Nambu-Goto and Polyakov Actions

If we denote the induced metric by

$$G_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu,$$

then the vanishing of the stress tensor says

$$G_{ab} - \frac{1}{2} h_{ab} G = 0$$

where $G = h^{ab} G_{ab}$ is the trace of the induced metric. Consider then

$$\det G_{ab} = \frac{1}{4} G^2 \det h$$

and so

$$\sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu = 2\sqrt{-\det G_{ab}}$$

Thus the Polyakov action (3.2) gives the Nambu-Goto action (3.1) when we integrate out the auxiliary metric.

3.2.2 Extending the Polyakov action

What other types of term could we add to the action?

- The obvious thing we could do is allow M to be a general Riemannian manifold with metric $g_{\mu\nu}(X)$. This makes the two-dimensional field theory on Σ highly non-linear and in practice difficult to analyse (we will however discuss this possibility later).

- Should we include an Einstein-Hilbert term? Since h_{ab} is appearing as a constraint, we do not want a kinetic term as the h_{ab} is not dynamical. What if we do it anyway and add in?

$$\frac{\lambda}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}R(h),$$

  where $R(h)$ is the Ricci scalar for the metric h_{ab}. This is just a Gauss-Bonnet term and is proportional to the Euler characteristic of the surface[6]

$$\chi = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}R.$$

- Additionally, we could consider adding a cosmological constant term

$$\Lambda\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}$$

  The equation of motion for the metric would then be

$$T_{ab} \sim -\Lambda h_{ab}$$

  Since $h^{ab}T_{ab} = 0$, we would conclude that

$$\Lambda h^{ab}h_{ab} = 0$$

  which is only acceptable if $\Lambda = 0$, so we will not consider cosmological constants on the worldsheet further.

- If we have other 'background' fields already living on M, we can pull them back to the worldsheet. A particularly important example is given by the two-form field

$$B = \frac{1}{2}B_{\mu\nu}(X)\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu,$$

  which when pulled back to Σ gives the contribution

$$-\frac{1}{2\pi\alpha'}\int_\Sigma B = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu B_{\mu\nu}(X).$$

[6]Such terms will in fact play a role when we consider constant Dilaton backgrounds. In fact, in two dimensions the Einstein tensor vanishes identically.

We will see later why such modifications to the action arise naturally in string theory.

Another possibility is a term of the form

$$\frac{1}{4\pi}\int_\Sigma R(h)\Phi(X),$$

where $R(h)$ is the worldsheet Ricci Scalar and $\Phi(X)$ is a spacetime scalar field, often called the dilaton. [7].

3.2.3 Symmetries of the Polyakov Action

The Polyakov action has a number of local and global symmetries that we must understand if we are successfully quantise the theory later on:

Global Symmetries: Poincare Invariance

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu, \qquad h_{ab} \to h_{ab}.$$

where $\Lambda^T = -\Lambda$ and $\Lambda \in SO(D-1,1)$ is a Lorentz transformation.

Local Symmetries: The theory is invariant under the local symmetries:

Reparameterizations Under the transformation $\sigma^a \to \sigma'^a(\sigma,\tau)$, the worldsheet fields transform as $X \to X'$, $h \to h'$ where

$$X'^\mu(\sigma',\tau') = X^\mu(\sigma,\tau), \qquad h_{ab}(\sigma,\tau) = \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}h'_{cd}(\sigma',\tau').$$

Under the infinitesimal transformation $\sigma^a \to \sigma^a - \xi^a(\sigma,\tau)$ the worldsheet fields transform infinitesimally as

$$\begin{aligned}\delta X^\mu &= \xi^a\partial_a X^\mu,\\ \delta h_{ab} &= \xi^c\partial_c h_{ab} + \partial_a\xi^c\, h_{bc} + \partial_b\xi^c\, h_{ac}\\ &= \nabla_a\xi_b + \nabla_b\xi_a,\\ \delta\sqrt{-h} &= \partial_a(\xi^a\sqrt{-h}).\end{aligned} \tag{3.3}$$

Weyl Transformations Weyl transformations are given by

$$X'^\mu(\sigma,\tau) = X^\mu(\sigma,\tau), \qquad h'_{ab}(\sigma,\tau) = e^{2\Lambda(\sigma,\tau)}h_{ab}(\sigma,\tau).$$

Infinitesimally,

$$\begin{aligned}\delta h_{ab} &= 2\Lambda h_{ab}\\ \delta X^\mu &= 0\end{aligned}$$

[7]This is not obviously Weyl-invariant; however, such a term can be included in a Weyl-invariant way in the quantum theory. The key to seeing something fishy is going on is to note that this term appears at a different order of α' to the other terms.

We will have to gauge fix the local symmetries in order to make sense of the quantum theory. There is a class of diffeomorphisms that can be cancelled out by cleverly chosen Weyl transformation. Thus, fixing the metric leaves a class of residual diffeomorphisms. We shall see that these generate the conformal group and will play a key role in the quantum theory.

3.3 Classical Solutions

We can use the three arbitrary degrees of freedom in $(\xi^a, \Lambda)$to fix the metric h_{ab}. We can use the diffeomorphsims to remove two degrees of freedom from the worldsheet metric and set it to be

$$h_{ab} = e^{2\phi}\eta_{ab}$$

where η_{ab} is the two dimensional Minkowski metric

$$\eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The action then becomes

$$S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma \left(- \dot{X}^2 + X'^2 \right)$$

where

$$\dot{X}^\mu := \partial_\tau X^\mu, \qquad X'^\mu := \partial_\sigma X^\mu.$$

This choice of metric is called *conformal gauge.* In conformal gauge we have $X^2 = X^\mu X_\mu$ and

$$\partial_a \left(\sqrt{h} h^{ab}\partial_b X^\mu\right) = \Box X^\mu = 0$$

where

$$\Box = -\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}$$

is the two dimensional D'Alembertian. The extrema of the action thus describes harmonic maps given by solutions of the form

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$$

Without loss of generality, we can express this in terms of Fourier modes as

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-in(\tau-\sigma)},$$

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}^\mu_n}{n}e^{-in(\tau+\sigma)},$$

For X^μ to be real, we require x^μ and p^μ to be real and

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu,$$

and similarly for $\bar{\alpha}_n^\mu$. We also define

$$\alpha_0^\mu = \bar{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu.$$

3.4 Classical Hamiltonian Dynamics

We stay now in the conformal gauge. We can define conjugate momentum

$$P_\mu = \frac{\delta S[X]}{\delta \dot{X}^\mu} = \frac{1}{2\pi\alpha'}\eta_{\mu\nu}\dot{X}^\nu.$$

And the vanishing of the stress tensor, like the mass-shell condition for the particle, must be imposed as a constraint.

One can define a Hamiltonian density in the usual way

$$\mathcal{H} = P_\mu \dot{X}^\mu - \mathcal{L} = \frac{1}{4\pi\alpha'}\left(\dot{X}^2 + X'^2\right)$$

where $\mathcal{L}$ is the Lagrangian density. Thus, the Hamiltonian is

$$H = \frac{1}{4\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma \left(\dot{X}^2 + X'^2\right).$$

As is standard in Hamiltonian mechanics, we introduce the Poisson bracket: Given functions on phase space $F(X,P)$ and $G(X,P)$, the Poisson bracket is defined as

$$\{F,G\}_{\text{PB}} = \int_0^{2\pi} \mathrm{d}\sigma \left(\frac{\delta F}{\delta X^\mu(\sigma)}\frac{\delta G}{\delta P_\mu(\sigma)} - \frac{\delta F}{\delta P_\mu(\sigma)}\frac{\delta G}{\delta X^\mu(\sigma)}\right),$$

which generalises the particle-like case

$$\{f,g\}_{\text{PB}} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

In particular $\{x,p\}_{\text{PB}} = 1$, which generalises in the field theory to

$$\{X^\mu(\tau,\sigma), P_\nu(\tau,\sigma')\}_{PB} = \delta^\mu_\nu \delta(\sigma-\sigma'),$$

which are the precursors to the canonical commutation relations. These give rise to corresponding Poisson bracket for the modes

$$\boxed{\{\alpha_m^\mu, \alpha_n^\nu\}_{PB} = -im\,\eta^{\mu\nu}\delta_{m+n,0}, \qquad \{\alpha_m^\mu, \bar{\alpha}_n^\nu\}_{PB} = 0 \qquad \{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\}_{PB} = -im\,\eta^{\mu\nu}\delta_{m+n,0}}$$

Let us briefly verify this statement. The commutator is at equal τ, so let us chose $\tau = 0$ for simplicity. Using the mode expansions

$$X^\mu(\sigma) = x^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\Big(\alpha^\mu_n e^{in\sigma} + \bar{\alpha}^\mu_n e^{-in\sigma}\Big),$$
$$P_\nu(\sigma') = \frac{p^\mu}{2\pi} + \frac{1}{2\pi}\sqrt{\frac{1}{2\alpha'}} \sum_{n\neq 0} \Big(\alpha^\mu_n e^{in\sigma} + \bar{\alpha}^\mu_n e^{-in\sigma}\Big)$$

The Poisson bracket is then

$$\begin{aligned}\{X^\mu(\sigma), P^\nu(\sigma')\} &= \frac{1}{2\pi}\{x^\mu, p^\nu\} \\ &+ \frac{i}{4\pi}\sum_{m,n\neq 0} \frac{1}{m}\Big(\{\alpha^\mu_m, \alpha^\nu_n\}_{PB}\, e^{i(m\sigma + n\sigma')} + \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\}_{PB}\, e^{-i(m\sigma+n\sigma')}\Big)\end{aligned}$$

Using the proposed Poisson brackets for the modes gives

$$\begin{aligned}\{X^\mu(\sigma), P^\nu(\sigma')\} &= \frac{\eta^{\mu\nu}}{2\pi} + \frac{\eta^{\mu\nu}}{2\pi}\sum_{m\neq 0} e^{im(\sigma-\sigma')} \\ &= \frac{\eta^{\mu\nu}}{2\pi}\sum_{m} e^{im(\sigma-\sigma')}\end{aligned}$$

where the first term has been absorbed into the sum in the last expression. Introducing the periodic delta-function;

$$\frac{1}{2\pi}\sum_m e^{im(\sigma-\sigma')} = \delta(\sigma - \sigma'),$$

we recover the correct canonical Poisson bracket

$$\{X^\mu(\sigma), P^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma').$$

3.4.1 The Classical Stress Tensor and the Wit Algebra

It is clearly sensible to introduce worldsheet light-cone coordinates

$$\sigma^\pm = \tau \pm \sigma.$$

We note that

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \,\mathrm{d}\sigma^2 = (\,\mathrm{d}\sigma^+,\,\mathrm{d}\sigma^-)\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}\mathrm{d}\sigma^+ \\ \mathrm{d}\sigma^-\end{pmatrix}$$

and

$$\partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right)$$

The action and equation of motion become

$$S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}\sigma^+\,\mathrm{d}\sigma^-\;\partial_+X\cdot\partial_-X, \qquad \partial_+\partial_-X^\mu = 0.$$

The stress tensor becomes

$$T_{++}(\sigma^+) = -\frac{1}{\alpha'}\partial_+X\cdot\partial_+X, \qquad T_{--}(\sigma^-) = -\frac{1}{\alpha'}\partial_-X\cdot\partial_-X$$

and T_{+-} vanishes identically (this is effectively the trace of T_{ab}).

We define the charges at $\tau = 0$ as

$$\boxed{\ell_n = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\;T_{--}(\sigma)e^{-in\sigma}, \qquad \bar{\ell}_n = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\;T_{++}(\sigma)e^{in\sigma}}$$

These are just the Fourier modes of the stress tensor components. We shall see that they are conserved on the constraint surface. Using

$$\partial_-X^\mu(\sigma^-) = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-in\sigma^-}, \qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu,$$

we find that

$$\begin{aligned}
\ell_n &= \frac{1}{2\pi\alpha'}\int_0^{2\pi}\mathrm{d}\sigma\;\partial_-X^\mu(\sigma)\partial_-X_\mu(\sigma)\\
&= \frac{1}{4\pi}\sum_{m,p}\alpha_m\cdot\alpha_p\int_0^{2\pi}\mathrm{d}\sigma\;e^{i(m+p-n)\sigma}\\
&= \frac{1}{4\pi}\sum_{m,p}\alpha_m\cdot\alpha_p 2\pi\delta_{p,n-m}
\end{aligned}$$

A similar result holds for $\bar{\ell}_n$, so that

$$\boxed{\ell_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \qquad \bar{\ell}_n = \frac{1}{2}\sum_m \bar{\alpha}_{n-m}\cdot\bar{\alpha}_m}$$

The constraint is then

$$\ell_m = 0, \qquad \bar{\ell}_n = 0.$$

These constraints represent the difference between two-dimensional massless Klein-Gordon theory and the bosonic string. In some sense they endow the two-dimensional theory on Σ with a target space interpretation.

One can show, using the Poisson brackets for the mode operators that these generators satisfy what is known as the Wit algebra

$$\boxed{\{\ell_m,\ell_n\} = -i(m-n)\ell_{m+n}, \qquad \{\ell_m,\bar{\ell}_n\} = 0, \qquad \{\bar{\ell}_m,\bar{\ell}_n\} = -i(m-n)\bar{\ell}_{m+n}.}$$

This is an infinite-dimensional Lie algebra. The subset $\ell_0, \ell_1, \ell_{-1}$ generate the sub-algebra $SL(2;\mathbb{R})$ and similarly for $\bar{\ell}_0, \bar{\ell}_1, \bar{\ell}_{-1}$. Together these generate $SL(2;\mathbb{C})$ - the mobius symmetry acting on the compactified conformal plane. We shall see later why this symmetry group is appearing.

The Hamiltonian may be written as

$$H = \frac{1}{2\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma\Big((\partial_+ X)^2 + (\partial_- X)^2\Big) = \frac{1}{2}\sum_n \Big(\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n\Big)$$

which we can write as

$$H = \ell_0 + \bar{\ell}_0.$$

This is the Hamiltonian of an infinite number of harmonic oscillators, each with Hamiltonian $H_n = \alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n$.

These generators generate a symmetry of the theory. The time evolution of the ℓ_n is given by the Poisson bracket with the Hamiltonian

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ell_n = \frac{\partial \ell_n}{\partial \tau} + \{\ell_n, H\} \propto n\ell_n \sim 0$$

We can understand this as follows: We consider the space of all embeddings X^μ and restrict to the space of physical embeddings in two stages. In the first, we impose the X^μ equations of motion, so that we are dealing with the space of harmonic maps into the target space $\mathscr{H}$. The $T_{ab} = 0$ condition is then imposed. We can think of the harmonic maps such that $T_{ab} = 0$ as a subspace, or *constraint surface* $\mathscr{N} \subset \mathscr{H}$. On the constraint surface $\mathscr{N}$, where $\ell_n = 0$, we see that $\dot{\ell}_n = 0$ and so the action of the Hamiltonian keeps ℓ_n within the constraint surface and so the ℓ_n are conserved charges. Noether's theorem tells us that we then expect an infinite-dimensional symmetry fo the theory. It is this infinite dimensional symmetry that is responsible for many of the miraculous features in string theory.

In summary: The worldsheet metric equation of motion is a constraint $T_{ab} = 0$. If we gauge fix h_{ab} to conformal gauge (or any other gauge) we cannot recover $T_{ab} = 0$ as an equation of motion and so it must be imposed as a constraint on the gauge-fixed theory.

Noether's theorem tells us that to each conserved quantity there is a an associated symmetry. In this case the conserved quantity is the stress tensor $T_{ab} = 0$. The symmetry associated with, or generated by, it is the conformal symmetry of the theory, i.e. that subset of Diff$\times$Weyl that is not fixed by fixing the metric. It is easy to see that a diffeomorphism

$$\sigma^a \to \sigma^a + v^a(\sigma),$$

will preserved the gauge choice for the metric if it can be undone by a Weyl transformation. These are the conformal transformations.

4 A first look at the quantum theory

Suppose that the phase space of the classical theory is $2d$-dimensional (we are counting each infinity of values of $X^\mu(\sigma)$ once), with coordinates $(X^\mu(\sigma), P_\mu(\sigma))$. We then have N constraints (again this is infinite-dimensional), giving a $2d - N$ dimensional constraint surface. On this surface each constraint gives a conserved charge ℓ_n and, by Noether's theorem, a symmetry. These n gauge symmetries reduce the physical phase space to be $2d - 2N$ dimensional. We could choose the physical phase space coordinates to be $(q^\mu(\sigma), \pi_\mu(\sigma))$ with an appropriate Poisson bracket.

How do we deal with these constraints in the quantum theory? There are two routes. Either we first reduce the classical theory to the $2d - 2n$ dimensional space and then quantise by endowing $(q^\mu(\sigma), \pi_\mu(\sigma))$ with canonical commutation relations. This is the track taken in the light-cone quantisation, where we solve the Virasoro constraints by going to light-cone coordinates and fixing a gauge there (and in the process breaking manifest space-time Lorentz invariance). One then choses to express the theory in $d - n$ dimensions using either the physical configuration or momentum coordinates. This gives rise to a theory with Hilbert space $\mathcal{H}_{\text{l.c.}}$.

In this course we choose a second route. We quantise the unconstrained variables $(X^\mu(\sigma), P_\mu(\sigma))$ by replacing the Poisson brackets by canonical commutation relations and then imposing the constraints on the Hilbert space and restricting to gauge-invariant states to give the physical Hilbert space $\mathcal{H}_Q$.

4.1 Canonical quantization

It is now a straightforward issue to quantise the theory. The standard approach to canonical quantization is to elevate the phase space variables to Hermitian operators and to replace Poisson brackets with commutators[8] in the *fundamental* relations[9]

$$\{\ .\ \}_{PB} \to -i[\ ,\ ].$$

In other words, we multiply our results for the Poisson-brackets by i to get the commutators. We define the worldsheet momentum and impose the canonical commutation (equal τ) relations

$$[X^\mu(\sigma), X^\nu(\sigma')] = 0, \qquad [P_\mu(\sigma), P_\nu(\sigma')] = 0, \qquad [P_\mu(\sigma), X^\nu(\sigma')] = -i\delta^\nu_\mu \delta(\sigma - \sigma').$$

Using the mode expansion

$$X^\mu(\sigma,\tau) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\Big(\alpha^\mu_n e^{-in(\tau-\sigma)} + \bar{\alpha}^\mu_n e^{-in(\tau+\sigma)}\Big)$$

[8]Remember, we have set $\hbar = 1$.

[9]Poisson brackets of more complicated functions of x and p may have complicated commutation relations involving higher powers of $\hbar$.

the corresponding commutation relations for the creation and annihilation operators are

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}, \qquad [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}, \qquad [\alpha^\mu_m, \bar{\alpha}^\nu_n] = 0.$$

We can construct other useful objects from these operators such as the Virasoro operators

$$L_m = \frac{1}{2}\sum_n \alpha_n \cdot \alpha_{m-n}, \qquad m \neq 0$$

These are the Fourier modes of the stress tensor but, as we shall see, they satisfy a slightly modified version of the Witt algebra, called the Virasoro algebra. As such it is useful to distinguish the L_n from their the classical counterparts ℓ_n. Rescaling the α^μ_n as

$$\alpha^\mu_n = \sqrt{n} a^\mu_n,$$

and recalling that the reality of the X^μ requires

$$(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n},$$

we see that the a^μ_n satisfy the algebra

$$[a^\mu_m, (a^\nu_n)^\dagger] = \delta_{m,n}\,\eta^{\mu\nu},$$

and so, for each spacetime direction μ, we have an infinite number of harmonic oscillators. The α^μ_n have the interpretation of creation (annihilation) operators for $n < 0$ ($n > 0$). What is it that they are creating? Left- and right-moving harmonic waves on the worldsheet. When we come to look at the worldsheet theory as a conformal field theory it will turn out to be more natural to work with the operators α^μ_n, rather than the a^μ_n, so we will stick with the α^μ_n.

We introduce the vacuum state[10] $|0\rangle$ and demand that

$$\boxed{\alpha^\mu_n|0\rangle = 0, \quad n \geq 0.}$$

We have to take care to make sense of this as an operator expression in these expressions. There is ambiguity in L_0 as α_n and α_{-n} do not commute unless $n = 0$ so we choose to define

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n.$$

We use : : to denote normal ordering and take the α_n with $n \leq 0$ ($n > 0$) to be creation (annihilation) operators. We can therefore sensibly define composite operators, such as

$$T_{--} = -\frac{1}{\alpha'} : \partial_- X \cdot \partial_- X : .$$

[10]Note that this is a vacuum state in the Hilbert space of string oscillations. It is not a vaccum of a spacetime theory as we have not discussed any means to create or destroy worldsheets, only the modes on them.

4.2 Physical State Conditions

We note that we may write $nN_n = \alpha_{-n} \cdot \alpha_n$, where N_n is the number operator[11] counting the quanta at level n and so we may write

$$L_0 = \frac{\alpha'}{4}p^2 + N, \qquad N = \sum_n nN_n, \qquad \bar{L}_0 = \frac{\alpha'}{4}p^2 + \bar{N}, \qquad \bar{N} = \sum_n n\bar{N}_n.$$

As usual, the constraint $T_{ab} = 0$, is too strong to impose as an operator constraint on the Hilbert space. Instead, in the spirit of Gupta-Bleuler quantisation of QED, we impose the weaker condition

$$L_n|\psi\rangle = 0, \qquad n > 0$$

for $|\psi\rangle$ to be a physical state. Hermiticity then imposes

$$\langle\psi|L_{-n} = 0, \qquad n > 0.$$

The issue of L_0 is a little subtle as α_n^μ and α_{-n}^μ do not commute so there is a potential normal ordering issue in imposing the constraint. The general statement is

$$L_0|\psi\rangle = a|\psi\rangle, \qquad \bar{L}_0|\psi\rangle = a|\psi\rangle$$

for some $a \in \mathbb{R}$ is expected. In later chapters, when we come to consider the BRST quantisation of the theory, we shall prove that $a = 1$ is required for a physically sensible theory. For now we shall assume $a = 1$.

In fact, instead of the conditions $L_0|\psi\rangle = |\psi\rangle$ and $\bar{L}_0|\psi\rangle = |\psi\rangle$, it is useful to consider $L_0^+|\psi\rangle = 2|\psi\rangle$ and $L_0^-|\psi\rangle = 0$, where

$$L_0^\pm = L_0 \pm \bar{L}_0.$$

The condition $L_0^-|\psi\rangle = 0$ ensures $N = \bar{N}$ in all physical states and is called the *level matching* condition. It is the only constraint that relates the left and right-moving sectors. The condition $(L_0^+ - 2)|\psi\rangle = 0$ is related to the spacetime equations of motion of the states, thus we are interested in the physical state conditions[12]

$$\boxed{(L_0^+ - 2)|\psi\rangle = 0, \qquad L_0^-|\psi\rangle = 0, \qquad L_n|\psi\rangle = 0 = \bar{L}_n|\psi\rangle, \qquad n > 0}$$

Just as the particle constraints gave the equation of motion in momentum space $p^2 + m^2 = 0$, these are the momentum space equations of motion for the physical excitations in string.

[11] Note that it is actually $a_n = \sqrt{n}\alpha_n$ that satisfies the operator algebr for a creation and annihilation operator.

[12] It turns out that, if $a = 1$ and $D = 26$, there are a number of states that have zero norm that satisfy these conditions. These sates decouple from all physical processes and so we really take the physical Hilbert space to be states of positive norm that satisfy the above conditions.

4.3 The spectrum

Let us explore the spectrum of the theory by considering the lowest lying states in the physical Hilbert space. This will give us a quantum theoretic description for the oscillations of the string.

4.3.1 The Tachyon

With no oscillators the most general state is a superposition of momentum eigenstates. A single momentum eigenstate is[13]

$$|k\rangle = e^{ik\cdot x}|0\rangle.$$

This describes the centre of mass motion of the string. We see the action of the centre of mass momentum operator is

$$p_\mu e^{ik\cdot x}|0\rangle = -i\frac{\partial}{\partial x^\mu}e^{ik\cdot x}|0\rangle = k_\mu|k\rangle,$$

where we have used the position space realisation of the commutator $\{x^\mu, p_\nu\} = i\delta^\mu_\nu$

A general superposition of such states may be written as

$$|T\rangle = \int \mathrm{d}k\; T(k)|k\rangle.$$

The condition $(L_0^+ - 2)|T\rangle = 0$ gives

$$\left(L_0^+ - 2\right)|T\rangle = \left(\frac{\alpha'}{2}p^2 + N + \bar{N} - 2\right)|T\rangle = \left(\frac{\alpha'}{2}k^2 - 2\right)|T\rangle = 0$$

The constraint thus gives the momentum space Klein-Gordon equation

$$\left(k^2 - \frac{4}{\alpha'}\right)T(k) = 0.$$

Comparing this with the momentum space Klein-Gordon equation $k^2 + M^2 = 0$, we see that the mass-shell condition is

$$p^2 + \frac{2}{\alpha'}\left(N + \bar{N} - 2\right) = 0.$$

We see this is simply a standard mass-shell condition with

$$M^2 = \frac{2}{\alpha'}\left(N + \bar{N} - 2\right).$$

And so for the lowest lying state we have

$$M^2 = -\frac{4}{\alpha'}$$

[13] We will see how this relates to a more familiar momentum eigenfunction $e^{ik\cdot X}$ later when we study the state operator correspondence.

The state is tachyonic. The other virasoro conditions do not place any further constraints on $T(k)$. It is interesting to note that the free Klein-Gordon action (expressed in terms of momentum space) may be given by $S[T] = \langle T|(L_0^+ - 2)|T\rangle$. This isn't quite right but it is not far from the truth.

The Tachyon is the main deficiency of the bosonic string. It leads to incurable problems in the theory. The supersymmetric string, about which we shall say little, provides a cure. The imposition of supersymmetry forces the ground state of the theory to be massless and so there is no tachyon. So why study the bosonic string? Many of the key ideas in superstring theory can be understood as relatively mild generalisations of what occurs in the bosonic string.

4.3.2 Massless States

Next we consider states of the form

$$|\varepsilon\rangle = \varepsilon_{\mu\nu}\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle$$

It is helpful to decompose the tensor $\varepsilon_{\mu\nu}$ into irreducible representations of the Lorentz group

$$\varepsilon_{\mu\nu} = \tilde{\phi}\eta_{\mu\nu} + \tilde{g}_{\mu\nu} + \tilde{b}_{\mu\nu}$$

where $\tilde{g}_{\mu\nu}(k)$ is symmetric and traceless and $\tilde{b}_{\mu\nu}(k)$ is antisymmetric and traceless.

We have satisfied the level matching constraint by construction but the conditions $L_1|\varepsilon\rangle = 0$, $\bar{L}_1|\varepsilon\rangle$, and $L_0|\varepsilon\rangle = \bar{L}_0|\varepsilon\rangle = |\varepsilon\rangle$ will impose additional physical constraints.

Let us first consider $L_1|\epsilon\rangle$. The key part is

$$\begin{aligned}
\frac{1}{2}\sum_n \alpha_{1-n}\cdot\alpha_n\,\varepsilon_{\mu\nu}\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle &= \alpha_1\cdot\alpha_0\;\varepsilon_{\mu\nu}\;\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle \\
&= k_\lambda\varepsilon_{\mu\nu}\bar{\alpha}^{\nu}_{-1}\alpha^{\lambda}_1\alpha^{\mu}_{-1}|k\rangle \\
&= k_\lambda\varepsilon_{\mu\nu}\bar{\alpha}^{\nu}_{-1}\Big([\alpha^{\lambda}_1,\alpha^{\mu}_{-1}] + \alpha^{\mu}_{-1}\alpha^{\lambda}_1\Big)|k\rangle \\
&= k_\lambda\varepsilon_{\mu\nu}\bar{\alpha}^{\nu}_{-1}\eta^{\lambda\mu}|k\rangle \\
&= k^\mu\varepsilon_{\mu\nu}\bar{\alpha}^{\nu}_{-1}|k\rangle
\end{aligned}$$

and so we require $k^\mu\varepsilon_{\mu\nu} = 0$. From the other conditions we find:

- $L_1|\varepsilon\rangle = 0$ implies $k^\mu\varepsilon_{\mu\nu}(k) = 0$
- $\bar{L}_1|\varepsilon\rangle = 0$ implies $k^\nu\varepsilon_{\mu\nu}(k) = 0$
- $L_0|\varepsilon\rangle = \bar{L}_0|\varepsilon\rangle = |\varepsilon\rangle$ implies $k^2 = 0$ and so the states are massless.

The physical states are then the graviton $|\tilde{g}\rangle$, the Kalb-Ramond or B-field $|\tilde{b}\rangle$ and the dilaton $|\tilde{\phi}\rangle$ given by

$$|\tilde{g}\rangle = \tilde{g}_{\mu\nu}\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle, \qquad |\tilde{b}\rangle = \tilde{b}_{\mu\nu}\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle$$

$$|\tilde{\phi}\rangle = \tilde{\phi}\alpha^{\mu}_{-1}\bar{\alpha}^{\mu}_{-1}|k\rangle$$

where $\tilde{g}_{\mu\nu}$ is symmetric and traceless and $\tilde{b}_{\mu\nu}$ is anti-symmetric.

These properties may be recovered form studying the linearised description of the spacetime action

$$S[\phi, g, B] = -\frac{1}{2\kappa^2}\int \mathrm{d}^D x\sqrt{-g}e^{-2\phi}\left(R - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + ...,$$

where $H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]}$, which is invariant under the gauge transformation $\delta B_{\mu\nu} = \partial_{[\mu}\lambda_{\nu]}$. The +... denote corrections of order α'. The Einstein-Hilbert action to quadratic order is the famous Fierz-Pauli action.

The split into Lorentz representations is motivated by our desire to understand these states as propagating states on spacetime. Is this the right thing to do? Should we be putting so much weight on a spacetime interpretation? T-duality suggests that the more natural object is the background tensor $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$. We shall say more about this later.

4.3.3 Massive States

We consider

$$\varepsilon_{\mu\nu}\alpha^{\mu}_{-2}\bar{\alpha}^{\nu}_{-2}|k\rangle + \varepsilon_{\mu\nu\lambda}\alpha^{\mu}_{-2}\bar{\alpha}^{\nu}_{-1}\bar{\alpha}^{\lambda}_{-1}|k\rangle + \bar{\varepsilon}_{\mu\nu\lambda}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\bar{\alpha}^{\lambda}_{-2}|k\rangle + \varepsilon_{\mu\nu\lambda\rho}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\bar{\alpha}^{\lambda}_{-1}\bar{\alpha}^{\rho}_{-1}|k\rangle$$

The mass of these states is

$$M^2 = \frac{4}{\alpha'}.$$

It is clear that the mass scales involved mean that, if we are to make contact with the Standard Model of particle physics, it must be solely through the massless sector, with masses generated in Higgs-like mechanisms, rather than directly from massive string excitations.

The physical state conditions will clearly require a careful treatment of L_2 and $\bar{L}_2$.

4.3.4 Spurious States and Gauge-Invariance

Part II

Path Integral Quantisation

So far we have only discussed free strings. In order to introduce interactions it is useful to have at our disposal more powerful techniques. As such we consider an alternative method of quantising the string - the path integral. If you have studied an advanced course of quantum field theory (such as Part III AQFT), path integrals will be familiar already. If they are new to you, do not worry! We will need only the simplest aspects of the path integral here.

5 The path integral

We begin with a crash course in path integrals in one-dimensional quantum field theory (quantum mechnics) before generalising the results to two-dimensional quantum field theory (string theory).

5.1 The Path Integral in Quantum Mechanics

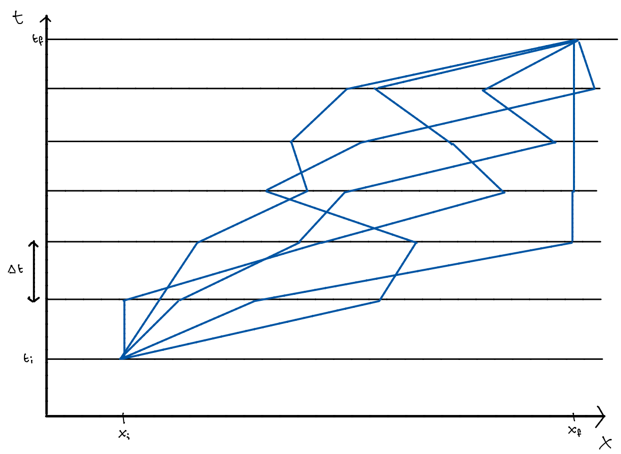


Figure 5. The path integral gives a weighted sum of all possible trajectories between the initial and final events.

We start with the transition amplitude, familiar from non-relatvistic quantum mechanics in one dimension in the Schrodinger picture

$$\langle x_f, t_f | x_i, t_i \rangle,$$

This is the amplitude associated with finding the particle at position x_i at time t_i and then finding at position x_f at a later time t_f. We can slice up the interval $[t_i, t_f]$

into $N+1$ equal units of duration Δt. We then look at the probabilities of following the path

$$(x_i,t_i)\to(x_1,t_1)\to(x_2,t_2)\to(x_3,t_3)...\to(x_N,t_N)\to(x_f,t_f),$$

where $t_n = t_i + n\Delta t$ and we integrate over all intermediate points, thus using a complete basis of states at each intermediate point

$$\langle x_f,t_f|x_i,t_i\rangle = \int \mathrm{d}x_1 ... \int \mathrm{d}x_N \; \langle x_f,t_f|x_N,t_N\rangle\langle x_N,t_N|x_{N-1},t_{N-1}\rangle...\langle x_1,t_1|x_i,t_i\rangle$$

We can factor off the time-dependence

$$\langle x_{j+1},t_{j+1}|x_j,t_j\rangle = \langle x_{j+1}|e^{iHt_{j+1}}e^{-iHt_j}|x_j\rangle$$

and, noting that $t_{j+1}-t_j=\Delta t$, we have

$$\begin{aligned}\langle x_{j+1},t_{j+1}|x_j,t_j\rangle &= \langle x_{j+1}|e^{-iH\,\Delta t}|x_j\rangle\\ &= \int \mathrm{d}p\,\mathrm{d}p'\langle x_{j+1}|p'\rangle\langle p'|e^{-iH\,\Delta t}|p\rangle\langle p|x_j\rangle\\ &= \int \mathrm{d}p\,\mathrm{d}p'\langle p'|e^{-iH\,\Delta t}|p\rangle e^{i(p'x_{j+1}-px_j)}\end{aligned}$$

where two complete basis of momentum states has been inserted in the second line and we have used the standard momentum wavefunction expression

$$\langle x_{j+1}|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ip\,x_{j+1}},$$

in the last line. We are interested in a free theory so $H=\hat{P}^2/2m$ and

$$\langle p'|e^{-i\hat{H}\,\Delta t}|p\rangle = \delta(p-p')e^{-iH(p)\,\Delta t}$$

where we have briefly introduced a hat on the Hamiltonian on the left hand side to note that it is an operator and a classical function on the right hand side. It is interesting to note that this is not canged by the inclusion of a potential $V(x)$ and, more generally, we have

$$\langle x_{j+1},t_{j+1}|x_j,t_j\rangle = \int \mathrm{d}p\;\exp\left(i\Delta t\left(\frac{x_{j+1}-x_j}{\Delta t} - H(p,\bar{x}_j)\right)\right)$$

where $\bar{x}$ is given by the average

$$\bar{x}_j = \frac{1}{2}(x_{j+1}+x_j).$$

And so

$$\langle x_f,t_f|x_i,t_i\rangle = \int\prod_{j=1}^{N}\mathrm{d}x_j\int\prod_{j=0}^{N}\mathrm{d}p_j\;\exp\left(i\Delta t\sum_{j=0}^{N}\left(p_j\;\frac{x_{j+1}-x_j}{\Delta t} - H(p_j,\bar{x}_j)\right)\right)$$

where $x_0 = x_i$ and $x_{N+1} = x_f$. We take the continuum limit

$$N \to \infty, \qquad \Delta t \to 0.$$

In this limit, p and x become functions of t and

$$\frac{x_{j+1} - x_j}{\Delta t} \to \dot{x}, \qquad \sum_{j=0}^{N} \Delta t \to \int_{t_i}^{t_f} \mathrm{d}t,$$

and so we have

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}x \int \mathcal{D}p \exp\left(i \int_{t_i}^{t_f} \mathrm{d}t \Big(p\dot{x} - H(p, x) \Big) \right)$$

where the functional integral notation

$$\lim_{N \to \infty, \Delta t \to 0} \int \prod_{j=1}^{N} \mathrm{d}x_j \equiv \int \mathcal{D}x,$$

has been introduced.

In many cases, we can perform the p_j integral before we take the limit to give the alternative, Lagrangian, expression

$$\boxed{\langle x_f, t_f | x_i, t_i \rangle = \mathcal{N} \int \mathcal{D}x \ \exp\left(\frac{i}{\hbar} S[x] \right)}$$

where the action is

$$S[x] = \int_{t_i}^{t_f} \mathrm{d}t \, L(x, \dot{x})$$

One may show that

$$\mathcal{N} = \lim_{N \to \infty} \left(\frac{m}{i\hbar \Delta t} \right)^{(N+1)/2},$$

which diverges in the limit; however, the physics will reside in a normalised version of this expression, so such factors will be consistently dropped. From now on we shall work in units in which $\hbar = c = 1$.

5.2 The Worldsheet Path Integral

We shall somewhat cavalierly assume that this final result generalises, even to those cases for which the steps in the derivation above do not hold. The justification will be that it works - the proof will be in the pudding. We could analyse this in more detail and satisfy ourselves that this is justified but our efforts will be required elsewhere. The obvious thing to do would be to discretise the σ coordinate on the worldsheet also

$$X_j^\mu(\sigma) \to X_{jk}^\mu$$

where k denotes the location on the lattice in the σ direction.

The path integral we need to make sense of is

$$\boxed{\langle \Psi_i | \Psi_f \rangle = \int_i^f \mathcal{D}X \mathcal{D}h \, e^{iS[X,h]}.}$$

The action is

$$S[X,h] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$$

We can improve matters slightly by Wick rotating to a Euclidean worldsheet so that $\mathrm{d}\tau\,\mathrm{d}\sigma \to i\,\mathrm{d}\tau\,\mathrm{d}\sigma$ so that the integral has a chance of converging. The use of a Euclidean worldsheet also has the benefit that we will be able to use the full power of complex analysis to perform worldsheet calculations later on.

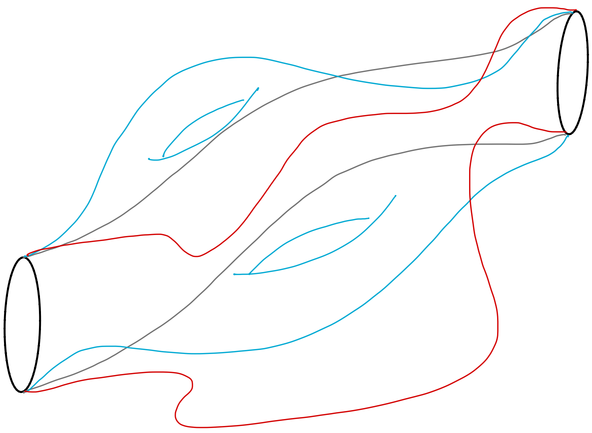


Figure 6. The path integral sums over all possible Riemann surfaces with give boundary conditions.

This clearly involves more than just cylinders.

Another problem is that the gauge symmetries mean that the path integral overcounts and does so infinitely. We would like to count only gauge-inequivalent configurations. Formally we may express this wish as

$$\langle \Psi_i | \Psi_f \rangle = \frac{1}{|\text{Weyl} \times \text{Diff}|} \int_i^f \mathcal{D}X \, \mathcal{D}h \, e^{-S[X,h]}.$$

We shall see that it is useful to incorporate initial and final asymptotic states as operator insertions into the path integral.

The main task in this chapter is to make sense of the object

$$\frac{\mathcal{D}X \, \mathcal{D}h}{\mathrm{dVol}(\text{Diff} \times \text{Weyl})}$$

This is hardly a well-defined object. What is it we are trying to capture in this expression? The space of diffeomorphisms is not simply connceted and we shall say something about 'large differomorphsis' later. For now, we will focus on only the connected component of Diff that contains the identity, which shall be denoted by Diff_0. Thus, we are interested in making sense of

$$\frac{\mathcal{D}X\ \mathcal{D}h}{\mathrm{dVol}(\mathrm{Diff}_0 \times \mathrm{Weyl})}$$

Of course, matters have not improved and this expression is also purely formal - $\int \mathcal{D}h$ is infinite and the diffeomorphism and Weyl groups are infinite dimensional. In order to make sense of what this means we need to explicitly decompose $\mathcal{D}h$ into metrics on the gauge slice and those that represent an over-counting given by the gauge symmetry

$$\mathcal{D}h = \mathcal{J} \times \mathcal{D}h_{\mathrm{phys}} \times \mathcal{D}\mathcal{V}$$

where $\mathcal{D}h_{\mathrm{phys}}$ is a measure on the space of *physically inequivalent* metrics. Thus, what we mean above is

$$\frac{1}{\mathrm{Vol}(\mathrm{Diff}_0 \times \mathrm{Weyl})}\int \mathcal{D}X\ \mathcal{D}h\ ... = \int \mathcal{J} \times \mathcal{D}h_{\mathrm{phys}}\ ...$$

and

$$\int \mathcal{D}\mathcal{V} = \mathrm{Vol}(\mathrm{Diff}{\times}\mathrm{Weyl}).$$

$\mathcal{J}$ is an appropriate Jacobian. In many ways, the aim of this section will be to find the form of the Jacobian $\mathcal{J}$. One can show (see the Appendix) that this Jacobian is a rational function of square-roots of functional determinants and cam be expressed as a functional integral, in much the same way a normal determinant may be expressed as a Gaussian integral over auxiliary variables

$$\sqrt{\frac{(2\pi)^n}{\det M}} = \int_V \mathrm{d}^n x\ e^{-\frac{1}{2}(x,Mx)}$$

for some self-adjoint operator M on an n-dimensional Euclidean space V. In the functional case the x's above are replaced by local fields and this Jacobian will contribute to the classical action.

A helpful trick in finding the measure on the space of metrics $\mathscr{H}$ is to instead find the measure on the tangent space to the space of metrics $T\mathscr{H}$. The Jacobean will be the same in either case. The measure on $T\mathscr{H}$ is given by $\mathcal{D}(\delta h)$ where

$$\mathcal{D}\delta h = \mathcal{J}\ \mathcal{D}(\delta_{\mathrm{phys}} h)\ \mathcal{D}(\delta_{\mathrm{Diff}\times\ \mathrm{Weyl}} h)$$

The general transformation of the metric is

$$\delta h_{ab} = \delta_t h_{ab} + \delta_\omega h_{ab} + \delta_v h_{ab}$$

where ω generates Weyl transformations and δ_v generates diffeomorphsims connected to the identity

$$\delta_\omega h_{ab} = 2\omega h_{ab}, \qquad \delta_v h_{ab} = \mathcal{L}_v h_{ab} = \nabla_a v_b + \nabla_b v_a.$$

It will be useful to extract the trace-ful part of the infinitesimal diffeomorphism and include that in the Weyl transformations so that

$$\omega \to \omega + \nabla^a v_a$$

and we define the trace-free diffeomorphsim

$$\boxed{(\mathcal{P}v)_{ab} \equiv \nabla_a v_b + \nabla_b v_a - h_{ab}\nabla^c v_c.}$$

so that

$$\delta h_{ab} = (\mathcal{P}v)_{ab} + 2(\omega + \nabla_c v^a)h_{ab}$$

We shall take all transformations that are not in $\text{Diff}_0 \times \text{Weyl}$ as physical. Given a particular metric $\hat{h}_{ab}$, can all other metrics be reached from this one using the above transformations? To answer this question we must deal with the fact that the path integral involves sums over all surfaces Σ subject to the boundary conditions, not just cylinders. As such, we need to learn a little more about two-dimensional geometry and topology.

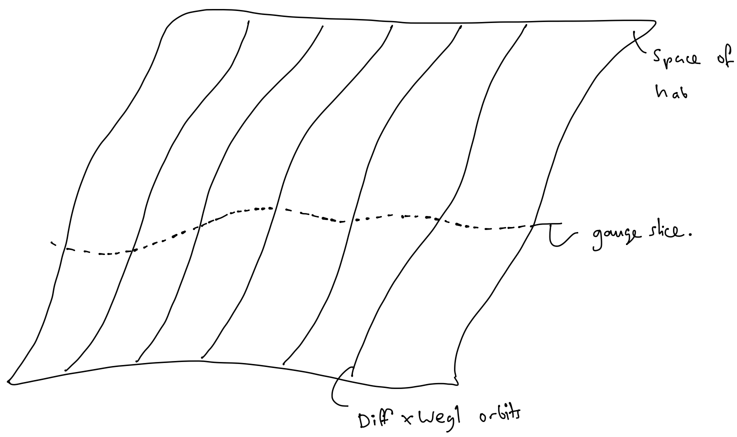


Figure 7. We want the path integral to count physically inequivalent metrics only once.

6 A crash course on Riemann Surfaces

Happily for us, mathematicians have long studied two dimensional Riemannian manifolds (Σ, h). By Riemannian geometry, we implicitly mean metrics, modulo diffeomorphisms. In the pantheon of Riemannian Geometry, the study of metrics defined, modulo Weyl invariance is particularly revered. Such manifolds are called *Riemann Surfaces*

$$\{\text{Riemannian Manifolds, mod Weyl }\} = \{\text{Riemann Surfaces}\}$$

Modulo diffeomorphisms is assumed in the definition of a Reimannian manifold. One of the many reasons why Riemann surfaces are so interesting is that, our manifold Σ looks locally like $\mathbb{C}$. If we require that these local patches are glued together using holomorphic transition functions we naturally have a Riemann surface.

6.1 Worldsheet Genus and Punctures

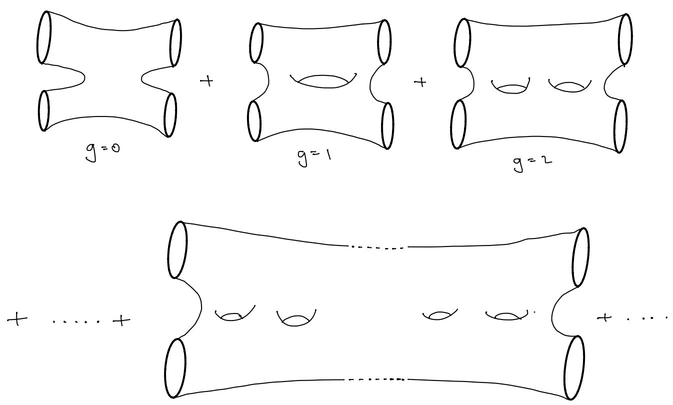
For Riemann surfaces without boundary, the topology of the surface is encoded in the Euler characteristic

$$\chi = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\, \sqrt{-h}R(h)$$

A more useful quantity is the *genus* g, which is related to the Euler characteristic by

$$\chi = 2 - 2g$$

From a perturbation theory perspective, it is clear that the genus counts the number of loops in a string diagram.



Another set of data we can include to specify a Riemann surface is the number and location of marked points, or *punctures* on the surface. In our considerations such punctures will be locations where we insert vertex operators and will correspond, by conformal transformation, to asymptotic states.

6.2 The Moduli Space of Riemann Surfaces

What is remarkable is that, for a given number of handles and boundaries, the space of inequivalent metrics on Riemann surfaces is finite-dimensional

$$\mathcal{M}_g = \frac{\{\text{metrics}\}}{\{\text{Diff} \times \text{Weyl}\}}$$

We call $\mathcal{M}_g$ the *moduli space* of Riemann surfaces. Note that it is the full Diff group used, not just that part connected to the identity.

An example of two tori not related by Weyl and diffeomorphsim invariance are sketched below

6.2.1 Example: T^2

One can use the Riemann-Roch theorem (or Atiyah-Singer index theorem) to show that the sphere has no moduli, i.e. on a sphere h_{ab} can be brought to a standard form globally using Diff and Weyl transformtions - given a standard round metric $\hat{h}_{ab}$ on S^2, all other metrics may be locally brought to the form

$$h_{ab} = e^{2\omega}\hat{h}_{ab}$$

using diffeomorphisms. Often this is phrased as all metrics on the sphere being conformally equivalent.

The torus is a little more interesting and provides an illustrative example. We shall construct the torus as a quotient of the complex plane by a discrete subgroup of translations. This can be written as

$$z \sim z + n\lambda_1 + m\lambda_2, \qquad n, m \in \mathbb{Z},$$

where λ_1 and λ_2 may be thought of as complex lattice vectors. λ_1 and λ_2 are not invariant under doffeomorphisms and Weyl transformations but their ratio is

$$\tau = \frac{\lambda_1}{\lambda_2}.$$

We can always define λ_1 and λ_2 such that $\text{Im}(\tau) > 0$. This object, not to be confused with the time coordinate on the worldsheet is often referred to as the complex structure. The torus inherits the flat metric from the complex plane and we could write the metric on the torus as

$$\mathrm{d}s^2 = |\,\mathrm{d}z + \tau\,\mathrm{d}\bar{z}|^2. \tag{6.1}$$

Put another way, it is always possible to bring a general metric satisfying the periodicity conditions to any form; however, one can always bring it to the form (6.1) for *some* $\tau \in$UHP, where UHP signifies the upper half plane

$$\text{UHP} = \{z \in \mathbb{C} | \text{Im}(z) > 0\}.$$

We take the upper half plane as the metric (6.1) is real so replacing τ with its complex conjugate is a symmetry and the torus degenerates if the real part of τ is allowed to go to zero. This is not quite the full story. If we write the periodicity condition as

$$z \sim z + n_a \lambda^a, \qquad n_a = (n, m), \qquad \lambda^a = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

we see the general form of the expression is invariant under

$$n_a \to (U^{-1})_a{}^b n_b, \qquad \lambda^a \to U^a{}_b \lambda^b$$

for $U \in SL(2)$. If $U \in SL(2;\mathbb{Z})$, then the components of n_a remain integers and we simply have another description of the same torus, thus the moduli space is

$$\mathcal{M}_1 = \frac{\text{UHP}}{SL(2;\mathbb{Z})}.$$

The $SL(2;\mathbb{Z})$ does not act freely on the upper half plane and there are fixed points which give rise to singularities in the quotient. The space is an *orbifold*.

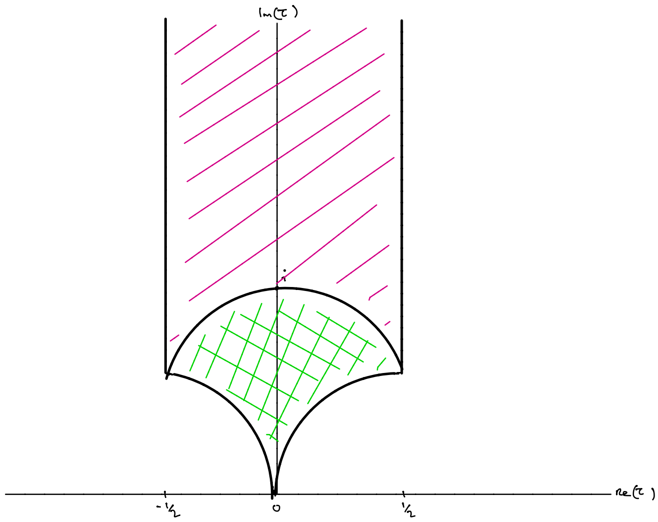


Figure 8. Under the identification with the modular group, the integration over moduli space may be take to be in the fundamental domain (shaded in purple).

6.2.2 The dimension of moduli space

The (real) dimension of the moduli space can be shown to be

$$s \equiv |\mathcal{M}_g| = \begin{cases} 0, & g = 0 \\ 2, & g = 1 \\ 6g - 6, & g \geq 2 \end{cases}$$

6.3 Moving around moduli space

Recall that

$$\delta_\omega h_{ab} = 2\omega h_{ab}, \qquad \delta_v h_{ab} = \mathcal{L}_v h_{ab} = \nabla_a v_b + \nabla_b v_a.$$

It will be useful to extract the trace-ful part of the infinitesimal diffeomorphism and include that in the Weyl transformations so that

$$\omega \to \omega + \nabla^a v_a$$

and we define the trace-free diffeomorphsim

$$\boxed{(\mathcal{P}v)_{ab} \equiv \nabla_a v_b + \nabla_b v_a - h_{ab}\nabla^c v_c.}$$

so that

$$\delta h_{ab} = (\mathcal{P}v)_{ab} + 2(\omega + \nabla_c v^a)h_{ab}$$

Introducing coordinates m^I on the finite dimensional moduli space $\mathcal{M}_g$ of Riemann surfaces, we may write

$$\delta h_{ab} = (\mathcal{P}v)_{ab} + 2(\omega + \nabla_c v^c)h_{ab} + \delta_t h_{ab}$$

where the moduli shift is

$$\delta_t h_{ab} = \delta m^I \frac{\partial}{\partial m^I} h_{ab} = \delta m^I \partial_I h_{ab}.$$

It will be useful to denote a vector in the tangent to the moduli space as $t^I = \delta m^I$. Note that we are really thinking of a metric on a Riemann surface as depending on the coordinates of the Riemann surface and the point on moduli space that selects that particular metric. As such it is helpful to write

$$h_{ab}(z,\bar{z}) \to h_{ab}(z,\bar{z},m^I)$$

explicitly to denote the fact that the metric is dependent on the point in moduli space.

The first thing to note is that there are trace contributions to both δ_{D_0} and δ_{Phys} are we have defined them. Such trace components can be absorbed into δ_ω and so we extract out the trace parts of each transformation and define

$$\delta_t h_{ab} \to t^I \mu_{Iab}$$

where

$$\mu_{Iab} := \frac{\partial h_{ab}}{\partial m^I} - \frac{1}{2} h_{ab} h^{cd} \frac{\partial h_{cd}}{\partial m^I}$$

is traceless and we incorporate the trace int the Weyl transformation. The Weyl transformation then becomes $\delta_\omega h_{ab} = 2\bar{\omega} h_{ab}$ where

$$\bar{\omega} := \omega + \nabla^a v_a + h^{ab} t^I \mu_{Iab}$$

6.4 Conformal Killing Vectors

Our plan is to gauge fix the diffeomorphisms by fixing the worldsheet metric h_{ab} to take some value $\hat{h}_{ab}$. This will not quite work as there are diffeomorphisms that are equivalent to a Weyl transformation and these will not be fixed by fixing the metric. In this section we wish to deal with those parts of Diff_0 that have overlap with Weyl. These are important as these are the symmetries that remain after we gauge fix the metric. We consider diffeomrophisms that may be undone by a Weyl transformation such that

$$\delta_{\text{CK}} h_{ab} = \nabla_a v_b + \nabla_b v_a + 2\omega h_{ab} = 0$$

taking the trace, we find

$$\omega = -\frac{1}{2}\nabla_a v^a,$$

and so we define

$$\boxed{(\mathcal{P}v)_{ab} = \nabla_a v_b + \nabla_b v_a - h_{ab}\nabla^c v_c},$$

and note that if $v_a \in \text{Ker}(\mathcal{P})$, i.e.

$$\nabla_a v_b + \nabla_b v_a - h_{ab}\nabla^c v_c = 0,$$

then v^a is a conformal Killing vector; i.e. if $v \in Ker(\mathcal{P}_1)$ then the diffeomorphism generated by v can be absorbed by a Weyl transformation.

We have

$$\text{Weyl} \cap \text{Diff}_0 = \text{CKV}$$

where the real dimension of the group generated by the CKVs is

$$\kappa \equiv |\text{CKV}| = \begin{cases} 6, \; g = 0 \\ 2, \; g = 1 \\ 0, \; g \geq 2 \end{cases}$$

Since these groups are finite dimensional, it is quite easy to deal with the overcounting in the path integral. A hint on how to do this comes from the Riemann sphere. At genus 0, Σ is the Riemann sphere. The CKG in this case is just $SL(2;\mathbb{C})$ - the Mobius group, which acts as

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1.$$

It is well known that we can fix such a transformation by specifying the mapping of three distinct points. We can chose a number of marked point on the Riemann surface and demand that they stay fixed under the action of the diffeomorphism group. In other words, we require that the vector fields v^a that generate the differomorphisms vanish (have fixed points) at three locations, which fixes three of the $\{a, b, c, d\}$ in the above transformation (the forth is fixed by requiring $ad - bc = 1$). This then means we are selecting a *particular* mobius transformation, not an integral over all possible transformations. It doesn't matter which points we choose as all choices are equivalent. Each point $\sigma^a \in \Sigma$ has two degrees of freedom. For the case of the sphere this means choosing three points (fixing six degrees of freedom) whilst for the torus, the group is $U(1) \times U(1)$ and we need only chose one point (two real degrees of freedom).

Without proof, we describe the CKVs in the two non-trivial cases.

6.4.1 The Sphere

There are three globally defined CKVs on the sphere, given in local coordinates by

$$\ell_{-1} = \partial_-, \qquad \ell_0 = \sigma^- \partial_-, \qquad \ell_1 = (\sigma^-)^2 \partial_-.$$

and

$$\bar{\ell}_{-1} = \partial_+, \qquad \bar{\ell}_0 = \sigma^+ \partial_+, \qquad \bar{\ell}_1 = (\sigma^+)^2 \partial_+.$$

These generate the conformal killing group $SL(2;\mathbb{C})$. One can show that these also give a subalgebra of the Virasoro algebra. These are the only vectrors

6.4.2 The Torus

In this case the two globally defined conformal killing vectors are the isometries

$$\ell_{-1} = \partial_-, \qquad \bar{\ell}_{-1} = \partial_+.$$

6.5 The Modular Group

Let us begin by pointing out that the diffeomorphism group is not simply connected - there are diffeomorphisms that are not connected to the identity. Denoting those diffeomorphisms that are connected to the identity by Diff_0, the modular group[14] is defined by

$$\mathscr{M} = \frac{\text{Diff}}{\text{Diff}_0}$$

What sort of things live in the modular group? In the case of the torus, the modular group is simply the same $SL(2;\mathbb{Z})$ we saw in our discussion of the moduli space.

All of the concrete calculations we do will be at tree level so the worldsheet will be topologically a sphere which has trivial modular group so this will not concern us further. It is worth noting that *modular invariance*, the invariance of physical observables under the action of the modular group is responsible for the finiteness of the string theory, order by order, in perturbation theory.

Notice that we can write

$$\mathcal{M}_g = \frac{\{\text{metrics}\}}{\{\text{Diff} \times \text{Weyl}\}} = \frac{\{\text{metrics}\}}{\{\text{Diff}_0 \times \text{Weyl}\}} \Big/ \mathscr{M}_g$$

[14] This is sometimes called the Mapping Class Group.

The space

$$\mathscr{T}_g = \frac{\{\text{metrics}\}}{\{\text{Diff}_0 \times \text{Weyl}\}},$$

is called the Teichmuller space.

Since the modular group may not act freely on $\frac{\{\text{metrics}\}}{\{\text{Diff}_0\}\times\{\text{Weyl}\}}$, this may not be a manifold as it may have isolated singularities. In general it is an *orbifold*. More on these later.

6.6 Summary

In Summary

- Not all diffeomorphisms are connected to the identity. The connected part including the identity is called Diff_0. $\text{Diff}/\text{Diff}_0 = \mathscr{M}_g$ (modular group).

- $\text{Diff}_0 \cap \text{Weyl}=\text{CKG}$, i.e. those diffeomorphisms that may be undone by a Weyl transformation. If $v \in Ker(\mathcal{P})$, then v is a CKV.

- The Teichmuller space is given by $\{\text{metrics}\}/\{\text{Diff}_0 \times \text{Weyl}\}$. The moduli space is given by $\mathcal{M}_g = \mathscr{T}_g/\mathscr{M}_g$

7 The Faddeev-Popov Determinant

We now come to the calculation of the Jacobian $\mathcal{J}$. An indirect way to calculate $\mathcal{J}$ is to use the Faddeev-Popov technique. We proceed by a finite-dimensional analogy and justify the procedure by the fact that the same result may be reached by alternative, more rigorous, methods which are outlined in the Appendix.

7.1 Faddeev-Popov on the sphere

To get the basic idea, let us consider the case where the worldsheet is topologically a sphere ($g = 0$). There are no moduli so we do not have to worry about large diffeomorphisms...

7.2 Faddeev-Popov with moduli

We now generalise to worldsheets of arbitrary genus g and with an arbitrary number of punctures n. The moduli space is then $\mathcal{M}_{g,n}$. Schematically we have

$$1 = \Delta_{FP}[\hat{h}, \hat{\sigma}] \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int_{\text{Diff}\times\text{Weyl}} \mathcal{D}U \; \delta[h^U - \hat{h}] \prod_i \delta(v(\hat{\sigma}_i))$$

where U denotes elements of the diffeomorphism and Weyl group connected to the identity and $\hat{h}_{ab}$ and h_{ab} are related by a gauge transformation and a change of moduli. The $\hat{\sigma}_i$ are the coordinates of $\kappa =$|CKV| punctures on the Riemann surface that are fixed by requiring the conformal Killing vectors to vanish at these points; $\delta v^a(\hat{\sigma}_i) = 0$. Fixing these κ points will completely fix the conformal killing symmetry. The t^I take values in the Teichmuller space, rather than the moduli space as we are yet to impose identifications under the modular group.

The Fadeev-Popov determinant is defined as

$$\boxed{1 = \triangle_{FP}[\hat{h}, \hat{\sigma}] \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int_{\text{Diff}\times\text{Weyl}} \mathcal{D}\omega \mathcal{D}v \, \delta\Big[h^{v,\omega} - \hat{h}\Big] \prod_{(a,i\in f)} \delta\Big(v^a(\hat{\sigma}_i)\Big)}$$

where the delta-functional is

$$\delta\Big[h_{v,\omega} - \hat{h}\Big] = \prod_{a,b,\sigma,\tau} \delta\Big(h^{v,\omega}_{ab}(\sigma,\tau) - \hat{h}_{ab}(\sigma,\tau)\Big).$$

The idea here is that we choose a particular metric $\hat{h}_{ab}$ and then generate another metric $h^{v,\omega}$ from this one by a gauge transformation connected to the identity

$$h^{v,\omega} = \hat{h}_{ab} + \delta h_{ab}, \qquad \delta h_{ab} = (\mathcal{P}v)_{ab} + 2\bar{\omega} h_{ab} + t^I \mu_{Iab}.$$

A useful choice will be $\hat{h}_{ab} = \eta_{ab}$ or, later when we work in Euclidean signature $\hat{h}_{ab} = \delta_{ab}$. We shall assume, without proof, that $\triangle_{FP}[\hat{h}, \hat{\sigma}]$ is invariant under infinitesimal gauge transformations.

We can use the functional version of the integral expression for the delta-function to write the delta functional as a functional integral. The inverse of the Fadeev-Popov determinant may then be written as

$$\begin{aligned}
\triangle^{-1}_{FP}[\hat{h}, \hat{\sigma}] &= \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int_{\text{Diff}\times\text{Weyl}} \mathcal{D}\omega \mathcal{D}v \, \delta[\delta h] \prod_{(a,i\in f)} \delta\Big(v^a(\hat{\sigma}_i)\Big) \\
&= \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int_{\text{Diff}\times\text{Weyl}} \mathcal{D}v \, \mathcal{D}\omega \, \mathcal{D}\beta \, \mathrm{d}^\kappa \zeta \, \exp\left(i(\beta|\mathcal{P}v + 2\bar{\omega}h + t^I \mu_I) + i \sum_{i=1}^{\kappa} \zeta^i_a v^a(\hat{\sigma}_i) \right)
\end{aligned}$$

where the delta functionals have been expressed as integrals over the auxiliary fields[15] β_{ab} and ζ^i_a, where $\beta_{ab} = \beta_{ba}$. Notice that, since the $v^a(\hat{\sigma}_i)$ are defined at *specific points*,

[15]Note that the β^{ab} here has nothing to do with the superpartner of the b-ghost.

the last term is an ordinary delta function, rather than a delta functional[16]. In the exponent we have introduced the inner product

$$(\beta|\mathcal{P}v + 2\bar{\omega}h + t^I\mu_I) \equiv \int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}\beta^{ab}\Big((\mathcal{P}v)_{ab} + 2\bar{\omega}h_{ab} + t^I\mu_{Iab}\Big)$$

for notational convenience.

The integration over the Weyl transformation parameters $\delta\bar{\omega} = \omega + ...$ can be done simply and imposes the constraint

$$\beta^{ab}h_{ab} = 0.$$

we then have

$$\triangle^{-1}_{FP}[\hat{h},\hat{\sigma}] = \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int_{\mathrm{Diff}\times\mathrm{Weyl}} \mathcal{D}v\,\mathcal{D}\beta\;\mathrm{d}^\kappa\zeta\;\exp\left(i\Big(\beta\Big|\mathcal{P}v + t^I\mu_I\Big) + i\sum_{i=1}^{\kappa}\zeta^i_a v^a(\hat{\sigma}_i)\right)$$

where it is understood that β^{ab} is traceless and symmetric.

We have an expression for $\triangle^{-1}_{FP}[\hat{h},\hat{\sigma}]$. We would like an expression for $\triangle_{FP}[\hat{h},\hat{\sigma}]$. It is a simple result in grassman integration that this may be achieved if we replace the bosonic variables with grassman ones. For example, in the finite-dimensional case, if we start with

$$\frac{1}{\det M} = \int_V \mathrm{d}\bar{z}\,\mathrm{d}z\;e^{-(\bar{z},Mz)}$$

If we introduce Grassmann variables θ, $\bar{\theta}$ then it is not too hard to show that

$$\int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\;e^{-(\bar{\theta},M\theta)} = \det M.$$

Thus changing the statistics of the auxiliary fields inverts the determinant.

To this end, we introduce the grassmann fields c^a, b^{ab}, η^i_a, and ξ^I as

$$v^a \to c^a, \qquad \beta_{ab} \to b_{ab}, \qquad \zeta^i_a \to \eta^i_a, \qquad t^I \to \xi^I$$

Notice that we are relating the conformal killing vectors to c ghosts and the deformations associated with the image of $\mathcal{P}$ to b ghosts.

We then have an expression for the Faddeev-Popov determinant

$$\triangle_{FP}[\hat{h},\hat{\sigma}] = \int \mathrm{d}^s\xi\,\mathcal{D}c\,\mathcal{D}b\;\mathrm{d}^\kappa\eta\;\exp\left(-(b|\mathcal{P}c + \xi^I\mu_I) + i\sum_{i=1}^{\kappa}\eta^i_a c^a(\hat{\sigma}_i)\right)$$

[16]We could of course write this as

$$\zeta^i_a v^a(\hat{\sigma}_i) = \int_\Sigma \mathrm{d}^2\sigma\,\delta^2(\sigma - \hat{\sigma}_i)\zeta^i_a(\sigma)v^a(\sigma),$$

where now ζ^i_a is promoted from being a Lagrange multiplier to a field.

Performing the ξ^I integral gives $\delta\left[(b|f_I h)\right]$, whilst doing the η_a^i integrals gives $\delta\left(c^a(\hat{\sigma})\right)$. The determinant then takes the form

$$\triangle_{FP}[\hat{h},\hat{\sigma}] = \int \mathcal{D}c\,\mathcal{D}b\,e^{-(b|\mathcal{P}c)}\prod_{I=1}^{s}\delta\Big[(b|\mu_I)\Big]\prod_{i=1}^{\kappa}\delta\Big(c^a(\hat{\sigma}_i)\Big)$$

We notice that, for a fermonic object $\delta(\theta)=\theta$, and so we can write

$$\boxed{\triangle_{FP}[\hat{h},\hat{\sigma}] = \int \mathcal{D}c\,\mathcal{D}b\,e^{iS[b,c]}\prod_{I=1}^{s}(b|\mu_I)\prod_{i=1}^{\kappa}\delta\Big(c^a(\hat{\sigma}_i)\Big)}$$

where the ghost action is

$$\boxed{S[b,c] = \frac{i}{4\pi}(b|\mathcal{P}c) = \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\hat{h}}\,b^{ab}(\mathcal{P}c)_{ab} = \frac{i}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\hat{h}}\,b^{ab}\nabla_a c_b}$$

where we have used the fact that b_{ab} is symmetric and traceless and have introduced a factor of 4π into the definition of the ghosts for future convenience.

We now return to the original path integral expression

$$\mathcal{Z}[0] = \frac{1}{|\mathrm{Diff}\times\mathrm{Weyl}|}\int \mathcal{D}h\,\mathcal{D}X\;e^{iS[h,X]}$$

and insert a factor of

$$1 = \triangle_{FP}(\hat{h},\hat{\sigma})\int_{\mathcal{M}_g}\mathrm{d}^s t\int_{\mathrm{Diff}\times\mathrm{Weyl}}\mathcal{D}\bar{\omega}\,\mathcal{D}v\,\delta[h_{v,\omega}-\hat{h}]\prod_{(a,i\in f)}\delta\left(v^a(\hat{\sigma}_i)\right)$$

to give

$$\begin{aligned}\mathcal{Z}[0] = {} & \frac{1}{|\mathrm{Diff}\times\mathrm{Weyl}|}\int_{\mathrm{Diff}\times\mathrm{Weyl}}\mathcal{D}\bar{\omega}\,\mathcal{D}v\prod_{a,i}\delta\Big(v^a(\hat{\sigma}_i)\Big)\int\mathcal{D}h\,\mathcal{D}X\int_{\mathcal{M}_g}\mathrm{d}^s t\,\delta[h_{v,\omega}-\hat{h}]\\ & \times\triangle_{FP}[\hat{h},\hat{\sigma}]\;e^{iS[h,X]}\end{aligned}$$

We now perform a gauge transformation

$$h_{v,\omega}\to h.$$

The action $S[h,X]$, measure $\mathcal{D}h$ and Faddeev-Popov determinant are all assumed invariant under such a transformation[17] so we have

$$\begin{aligned}\mathcal{Z}[0] = {} & \frac{1}{|\mathrm{Diff}\times\mathrm{Weyl}|}\int_{\mathrm{Diff}\times\mathrm{Weyl}}\mathcal{D}\bar{\omega}\,\mathcal{D}v\prod_{a,i}\delta\Big(v^a(\hat{\sigma}_i)\Big)\\ & \times\int_{\mathcal{M}_{g,n}}\mathrm{d}^s t\int\mathcal{D}X\,\mathcal{D}h\,\delta[h-\hat{h}]\;\triangle_{FP}[\hat{h},\hat{\sigma}]\;e^{iS[h,X]}\end{aligned}$$

[17] One can show that this is true iff $D=26$.

where the terms in the second line are all independent of v and ω. The delta functional kills the metric integral and sets $h_{ab} = \hat{h}_{ab}$, leaving

$$\mathcal{Z}[0] = \frac{1}{|\text{Diff} \times \text{Weyl}|} \int \mathcal{D}X \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \left(\int_{\text{Diff}\times\text{Weyl}} \mathcal{D}\bar{\omega}\, \mathcal{D}v \prod_{a,i} \delta\Big(v^a(\hat{\sigma}_i)\Big) \right) e^{iS[\hat{h},X]} \triangle_{FP}(\hat{h}, \hat{\sigma})$$

We notice that the terms in brackets factor out of the functional integrals and may be identified as

$$\int_{\text{Diff}\times\text{Weyl}} \mathcal{D}\bar{\omega}\, \mathcal{D}v \prod_{a,i} \delta\Big(v^a(\hat{\sigma}_i)\Big) = \frac{|\text{Diff} \times \text{Weyl}|}{|\text{CKG}|},$$

where we recall that the delta functions remove the CKV part of Diff_0. We therefore have the gauge fixed expression

$$\mathcal{Z} = \frac{1}{|\text{CKG}|} \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int \mathcal{D}X\, e^{iS[\hat{h},X]} \triangle_{FP}(\hat{h}, \hat{\sigma})$$

We note that, since the c^a are grassman, we can write

$$\delta\Big(c^a(\hat{\sigma}_i)\Big) = \prod_{a=1,2} c^a(\hat{\sigma}_i)$$

Putting in our expression for the Faddeev-Popov determinant

$$\boxed{\mathcal{Z} = \frac{1}{|\text{CKG}|} \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int \mathcal{D}X\, \mathcal{D}c\, \mathcal{D}b\, e^{iS[X,\hat{h},b,c]} \prod_{I=1}^{s} (b|\mu_I) \prod_{i,a} c^a(\hat{\sigma}_i)}$$

where the action is the sum of matter and ghost terms.

$$S = S[\hat{h}, X] + S[b, c, \hat{h}]$$

Notice that, in the case of the sphere, the moduli are those of $n-3$ punctures - three fixed by the conformal Killing transformations, so we effectively have

$$\int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \prod_{i,a} c^a(\hat{\sigma}_i) \to \int \prod_{i=1}^{n-3} \mathrm{d}^2 z_i.$$

We will see this play an important role when we consider scattering amplitudes.

7.3 UV finiteness

The moduli space integral is over a finite domain, often called the fundamental domain. This a finite space in the one-loop amplitude and is given by

$$\text{UHP}/PSL(2;\mathbb{Z})$$

where UHP denotes the upper half plane in $\mathbb{C}$. The modular group makes this finite so there are no ultraviolet divergences as seen if we were to attempt to quantise the Einstein-Hilbert action.

7.4 Ghosts!

It is sometimes useful to keep the gauge-fixing part explicit and write

$$\mathcal{Z} = \frac{1}{|\mathrm{CKG}|} \int_{\mathcal{M}_{g,n}} \mathrm{d}^s t \int \mathcal{D}X \mathcal{D}h\, \mathcal{D}c\, \mathcal{D}b\, e^{iS[X,\hat{h},b,c]}\, \delta[h - \hat{h}] \prod_{I=1}^{s} (b|\mu_I) \prod_{i,a} c^a(\hat{\sigma}_i)$$

On the support of this delta-functional we can exchange h_{ab} and $\hat{h}_{ab}$. It will also be useful to incorporate the delta-functional into the action. To this end, we introduce the symmetric, traceless fields B^{ab} and introduce

$$S_{\mathrm{gf}}[B, h] = \frac{1}{4\pi} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} B^{ab} \Big(\delta_{ab} - h_{ab} \Big),$$

to the action. Functional integration over B^{ab} gives the gauge-fixing condition $h_{ab} = h_{ab}$. The full action is now

$$\begin{aligned} S[X, h, b, c, B] = &-\frac{1}{4\pi\alpha'} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{i}{2\pi} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} b_{ab} \nabla^a c^b \\ &+ \frac{1}{4\pi} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} B^{ab} \Big(\delta_{ab} - h_{ab} \Big) \end{aligned} \tag{7.1}$$

The quantization of the ghosts follows from this action and the fact that the are Grassmann fields. The key point is that $b_{ab}(\tau, \sigma)$ and $c^a(\tau, \sigma)$ are conjugate fields. It is worth pointing out that the ghosts have integer spin but fermionic statistics. They are not physical observables and this violation of the spin-statistics theorem is the hallmark of ghost fields.

Part III
Conformal Field Theory

8 The worldsheet theory as a Conformal Field Theory

We have seen in the previous section that the path integral can be made sense of by the introduction of a Fadeev-Popov ghost system. The resulting action is then

$$S[X,b,c] = T\int_\Sigma \eta_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + b\bar{\partial}c + \bar{b}\partial\bar{c}$$

We may take this as the starting point for our gauge-fixed theory and

8.1 The Conformal Plane

It will be useful to Wick rotate to a Euclidean worldsheet and write the theory in terms of the complex plane with coordinates $(z,\bar{z})$ using the conformal transformation

$$z = e^{\tau - i\sigma}, \qquad \bar{z} = e^{\tau + i\sigma}.$$

Consider a string propagating from the infinite past $\tau = -\infty$, this maps to a string propagating from the origin $(z,\bar{z}) = 0$ radially outwards. Thus the dilation operator L_0^+ plays the role of the worldsheet Hamiltonian and time ordering becomes radial ordering.

9 Introduction to CFTs

A conformal transformation on flat space preserves the angle between any two straight lines. In particular, under $x^\mu \to x'^\mu(x)$, the metric transforms as

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x)\eta_{\mu\nu}.$$

for some function $\Lambda(x)$. Note that if $\Lambda(x) = 1$ this defines the Poincare group of transformations[18]. More generally, on a curved spacetime, we have

$$g'_{\rho\sigma}(x')\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x)g_{\mu\nu}(x).$$

If we introduce an infinitesimal parameter $\epsilon << 1$ and write the transformation as

$$x^\mu \to x'^\mu(x) = x^\mu + \epsilon v^\mu(x) + \mathcal{O}(\epsilon^2),$$

[18] Lorentz transformations and spacetime translations, sometimes written as $ISO(d-1,1)$.

then to first order we have on flat space

$$\begin{aligned}\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} &= \eta_{\rho\sigma}\left(\delta^{\rho}_{\mu}+\epsilon\partial_{\mu}v^{\rho}+...\right)\left(\delta^{\sigma}_{\nu}+\epsilon\partial_{\nu}v^{\sigma}+...\right)\\ &= \eta_{\mu\nu}+\epsilon\left(\partial_{\mu}v_{\nu}+\partial_{\nu}v_{\mu}\right)+...\\ &= \Lambda(x)\eta_{\mu\nu}, \end{aligned} \tag{9.1}$$

where the ellipsis denote terms of order ϵ^2 and higher. If we write $\Lambda(x) = e^{\epsilon\omega(x)}$ for some function $\omega(x)$, then we can write

$$\eta_{\mu\nu}+\epsilon\left(\partial_{\mu}v_{\nu}+\partial_{\nu}v_{\mu}\right)+... = \eta_{\mu\nu}+\epsilon\omega(x)\eta_{\mu\nu}+...$$

and so to leading order

$$\partial_{\mu}v_{\nu}+\partial_{\nu}v_{\mu} = \omega(x)\eta_{\mu\nu}.$$

We can find $\omega(x)$ in terms of $v^{\mu}(x)$ by first taking the trace of the above equation to give

$$\omega(x) = \frac{2}{d}\partial\cdot v(x),$$

where d is the dimension of the spacetime in question. Substituting this back in to eliminate $\omega(x)$ gives that the condition that the $v^{\mu}(x)$ generate infinitesimal conformall transformations is that they satisfy

$$\boxed{\partial_{\mu}v_{\nu}+\partial_{\nu}v_{\mu} = \frac{2}{d}\eta_{\mu\nu}\partial_{\lambda}v^{\lambda}(x).} \tag{9.2}$$

9.1 The special case of $d = 2$

Our worldsheet is two-dimensional. We shall see that soemthing magical happens when we restrict to the special case of $d = 2$ which allows the full power of complex analysis to be brought to bear. We choose the Euclidean metric to be

$$h_{\mu\nu} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and choose coordinates $x^{\mu} = (\tau,\sigma)$. The equation (9.2) places constraints on the components v_{τ} and v_{σ}. It is easy to show that, with this choice of metric (9.2) becomes the pair of equations

$$\frac{\partial v_{\tau}}{\partial\tau} = \frac{\partial v_{\sigma}}{\partial\sigma}, \qquad \frac{\partial v_{\sigma}}{\partial\tau} = -\frac{\partial v_{\tau}}{\partial\sigma}.$$

But these are nothing more that the Cauchy-Riemann equations for the complex function $v = v^{\tau}+iv^{\sigma}$. If we introduce the complex coordinates

$$\omega = \tau+i\sigma, \qquad \bar{\omega} = z = \tau-i\sigma,$$

then the condition for the vector field to generate infinitesimal conformal transformations is that the complex function $v = v^\tau + iv^\sigma$ is holomorphic; i.e.

$$\bar{\partial} v = 0.$$

This result is clearly special to $d = 2$ and the fact that any holomorphic function $\omega \to z = f(\omega)$ gives a conformal transformation suggests that there are an infinite number of generators of the conformal symmetry.

A particularly useful choice of coordinates are

$$z = e^{\tau + i\sigma}, \qquad \bar{z} = e^{\tau - i\sigma}.$$

Under this map the cylinder is mapped to the Riemann sphere. In particular, the infinite past is mapped to the origin and the infinite future to the point at infinity and time[19] evolution becomes radial evolution.

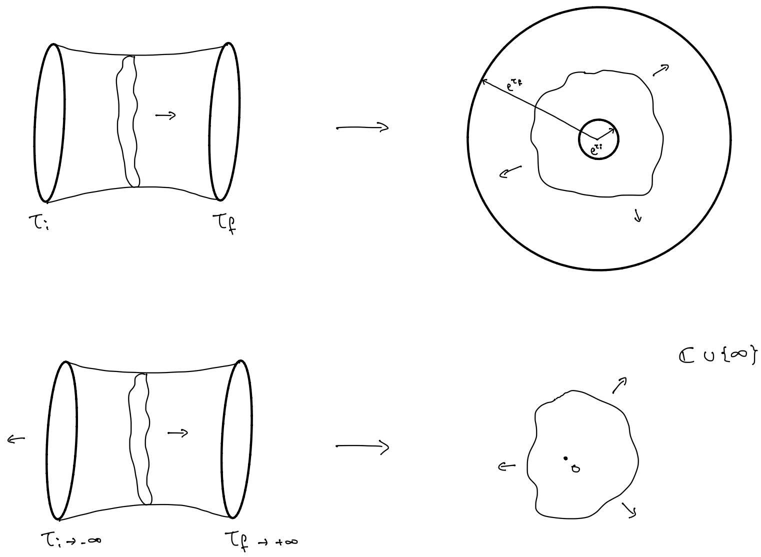


Figure 9.

9.2 The Witt Algebra

We can identify a basis for these generators as follows. We expand the (holomorphic) vector field as

$$v(z) = \sum_n v_n z^{n+1}$$

where the v_n are constants. Under the conformal transformation, we have[20]

$$z \to z + v(z) = z + \sum_n v_n z^{n+1}.$$

[19]Of course what we mean by 'time' here is Euclidean time, which is Wick rotated from our usual notion of worldsheet time.

[20]We henceforth absorb the infinitesimal parameter ϵ into the definition of $v(z)$.

This transformation is generated by[21]

$$-\sum_n \ell_n v_n$$

where

$$\ell_n = -z^{n+1}\partial_z$$

and similarly for $\bar{\ell}_n$. Since n takes any integer value, there are a countably infinite number of such generators and they close to give

$$\begin{aligned}[\ell_n, \ell_m] &= [-z^{n+1}\partial_z, -z^{m+1}\partial_z] \\ &= -(m-n)z^{m+n+1}\partial_z, \end{aligned} \tag{9.3}$$

and so

$$\boxed{[\ell_n, \ell_m] = (m-n)\ell_{m+n}.}$$

This is called the Witt algebra.

9.3 Conformal Fields

First, some definitions:

Chiral Field is a field that depends only on z; $\Phi(z)$. Similarly an anti-chiral field depends only on $\bar{z}$. We also use holomorphic/anti-holomorphic and, in the context of a Minkowski worldsheet metric it makes sense to speak of left/right-movers as the embedding feilds satisfy the wave equation.

Conformal Dimension refers to how a field transforms under conformal transformations. If a field transforms as

$$\Phi(z,\bar{z}) \to \Phi'(z',\bar{z}') = \lambda^h \bar{\lambda}^{\bar{h}} \Phi(\lambda z, \bar{\lambda}\bar{z})$$

under $(z,\bar{z}) \to \lambda z, \bar{\lambda}\bar{z}$, then we say the field has conformal dimension $(h, \bar{h})$. Note that a chiral field has dimension $(h, 0)$.

Primary Field Under the holomorphic transformation $z \to z' = f(z)$, a Primary conformal field of weight $(h, \bar{h})$ transforms as[22]

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial f}{\partial z}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z}))$$

[21] The negative sign is for later convenience.

[22] If $f(z)$ is in $SL(2;\mathbb{C})/\mathbb{Z}_2$ (Moebius) then we call this a quasi-primary field.

The infinitesimal transformation of primary field may be found by expanding in powers of ϵ as we did with the metric. Let $f(z) = z + \epsilon v(z) + ...$, then

$$\begin{aligned}\left(\frac{\partial f}{\partial z}\right)^h &= (z + \epsilon v(z) + ...)^h = 1 + h\partial_z v(z) + ... \\ &= \Phi(z,\bar{z}) + v(z)\partial_z\Phi(z,\bar{z}) + ... \end{aligned} \tag{9.4}$$

Also

$$\begin{aligned}\Phi(f(z),\bar{z}) &= \Phi(z + v(z) + ..., \bar{z}) \\ &= \Phi(z,\bar{z}) + v(z)\partial_z\Phi(z,\bar{z}) + ... \end{aligned} \tag{9.5}$$

and so

$$\begin{aligned}\Phi(z,\bar{z}) \to & \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial f}{\partial z}\right)^{\bar{h}} \Phi(f(z),\bar{f}(\bar{z})) \\ = & \; (1 + h\partial v(z))(1 + \bar{h}\bar{\partial}\bar{v}(\bar{z}))(\Phi(z,\bar{z}) + v(z)\partial\Phi(z,\bar{z}) + \bar{v}(\bar{z})\bar{\partial}\Phi(z,\bar{z})) + ... \\ = & \; \Phi(z,\bar{z}) + (h\partial v(z) + v(z)\partial + \bar{h}\bar{\partial} + \bar{v}\bar{\partial})\Phi(z,\bar{z}) + ... \end{aligned} \tag{9.6}$$

and so, to first order, the conformal transformation of a primary field is given by

$$\boxed{\delta_{v,\bar{v}}\Phi(z,\bar{z}) = \Big(h\partial v(z) + v(z)\partial + \bar{h}\bar{\partial} + \bar{v}\bar{\partial}\Big)\Phi(z,\bar{z})}$$

10 Conformal Transformations from the stress tensor

Another important (perhaps the most important) field is the stress tensor T_{ab}. We saw the stress tensor defined in terms of the functional derivative of the action with respect to the world-sheet metric and the requirement $T_{ab} = 0$ arose as a constraint. In a way that we will study later, we will see that this same field generates the conformal transformations. The conformal symmetry holds at the classical level but we shall see the emergence of potential anomalies when we come to the quantum theory later and as such there are additional associated subtleties when we deal with the stress tensor as a quantum field. For now we will confine our attention to the classical field theory.

10.1 The Stress Tensor and Noether's Theorem

In the classical theory, Noether's theorem tells us that a symmetry gives rise to a conserved current j^a, so that for some action

$$S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\; \partial_a X^\mu \partial_a X^\nu \eta_{\mu\nu},$$

under the transformation

$$\delta_v X^\mu = v^a \partial_a X^\mu,$$

we have

$$\delta_v S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma \left((\partial_a v^b)\,\partial_b X^\mu \partial_a X_\mu + v^b\,\partial_a(\partial_b X^\mu)\partial_a X_\mu\right)$$
$$= \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\,(\partial^a v^b)T_{ab},$$

where T_{ab} is the stress tensor.[23]

Noether's theorem tells us that this symmetry $\delta_v S[X] = 0$ implies the existence of a conserved current

$$\delta_v S[X] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\,(\partial^a v^b)T_{ab} = -\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\, v^b(\partial^a T_{ab}) = 0$$

and so we have

$$\boxed{\partial^a T_{ab} = \partial^a T_{ba} = 0}$$

where we have used the fact that the stress tensor is symmetric.

The conserved charge, evaluated at $\tau = 0$ is then

$$Q = Q_+ + Q_-, \qquad Q_\pm = \frac{1}{2\pi}\oint_0^{2\pi} \mathrm{d}\sigma\; v^\pm(\sigma)T_{\pm\pm}(\sigma)$$

The symmetry transformation is then generated by the commutator with the Poisson bracket

$$\delta_v X^\mu = \{Q, X^\mu\}_{\mathrm{PB}}.$$

Something similar happens in the quantum theory.

10.2 Complex Coordinates

Using complex coordinates we find that

$$T_{zz} = \partial X^\mu \partial X_\mu, \qquad T_{\bar{z}\bar{z}} = \bar{\partial} X^\mu \bar{\partial} X_\mu, \qquad T_{z\bar{z}} = 0 = T_{\bar{z}z}.$$

From now on we shall refer to the non-trivial components of the stress tensor by the shorthand

$$T := T_{zz}, \qquad \bar{T} := T_{\bar{z}\bar{z}}.$$

The conservation law simply states that $\bar{\partial}T = 0$ and $\partial\bar{T} = 0$; i.e. $T(z)$ and $\bar{T}(\bar{z})$ are holomorphic and anti-holomorphic respectively.

Under the change of coordinates

$$\mathrm{d}\tau\,\mathrm{d}\sigma = -\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2i|z|^2}, \qquad \partial_\tau = z\partial + \bar{z}\bar{\partial}, \qquad \partial_\sigma = iz\partial - i\bar{z}\bar{\partial},$$

[23]Notice that, since the stress tensor is traceless, we can write this as

$$\delta_v S[X] = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,(\mathcal{P}v)^{ab}T_{ab}.$$

so that the action becomes

$$S[X] = \frac{i}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \; \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu .$$

We choose to redefine the action $S[X] \to iS[X]$, so that in the path integral

$$\int \mathcal{D}X \; e^{iS[X]} \to \int \mathcal{D}X \; e^{-S[X]},$$

where

$$S[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \; \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu .$$

Moving from Minkowski to Euclidean space must be handled carefully; however, the subtleties involved will not arise in the limited cases we will study in this course and so we shall take a rather cavalier attitude of moving between Minkowski and Euclidean spaces whenever the need arises.

10.3 Ward Identities and Conformal Transformations

We investigate the quantum version of Noether's theorem. Classically, if we have a Lagrangian with fields ϕ and make some change of the form $\delta\phi = \varepsilon f(\phi, \partial\phi, ...)$ with parameter ε that preserves the action, then this is a classical symmetry of the theory if ε is a constant.

For example, if we have the action

$$S[X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \partial_a X^\mu \partial^a X_\mu ,$$

then under the rigid transformation $\delta X^\mu = \varepsilon^a \partial_a X^\mu$, the action is invariant. If we allow ε to depend on the coordinates on Σ, we find

$$\delta S[X] = \frac{1}{2\pi} \int_\Sigma \mathrm{d}^2\sigma \, (\partial^a \varepsilon^b) T_{ab}$$

where T_{ab} is the stress tensor so, classically, the invariance of he action under the transformation implies the conservation of the stress tensor

$$\partial^a T_{ab} = 0.$$

Let us look at the quantum analogue of Noether's theorem. Consider the correlation function

$$\langle \phi(z_1)...\phi(z_n) \rangle = \int \mathcal{D}\phi \; e^{-S[\phi]} \; \phi(z_1)...\phi(z_n).$$

Consider a small change in the field

$$\phi' = \phi + \delta\phi, \qquad S[\phi'] = S[\phi] + \delta S$$

Then

$$\begin{aligned}
\langle \phi'(z_1)...\phi'(z_n)\rangle &= \int \mathcal{D}\phi'\ e^{-S[\phi]-\delta S}\ \phi'(z_1)...\phi'(z_n) \\
&= \int \mathcal{D}\phi\ \Big(1-\delta S[\phi]+...\Big)\times\Big(\phi_1+\delta\phi_1+...\Big)\times ... \times\Big(\phi_1+\delta\phi_1+...\Big) \\
&= \int \mathcal{D}\phi\ e^{-S[\phi]}\ \phi_1...\phi_n - \int \mathcal{D}\phi\ e^{-S[\phi]}\ \delta S[\phi]\ \phi_1...\phi_n \\
&\quad + \int \mathcal{D}\phi\ e^{-S[\phi]} \sum_{k=1}^{n} \phi_1...\phi_{k-1}\ \delta\phi_k\ \phi_{k+1}...\phi_n \\
&= \langle \phi_1...\phi_n\rangle - \langle \delta S[\phi]\ \phi_1...\phi_n\rangle + \sum_{k=1}^{n}\langle \phi_1...\phi_{k-1}\ \delta\phi_k\ \phi_{k+1}...\phi_n\rangle
\end{aligned}$$

where we have assumed $\mathcal{D}\phi' = \mathcal{D}\phi$. If we define

$$\delta\langle \phi(z_1)...\phi(z_n)\rangle = \langle \phi'(z_1)...\phi'(z_n)\rangle - \langle \phi(z_1)...\phi(z_n)\rangle = 0,$$

then we have

$$\langle \delta S[\phi]\ \phi_1...\phi_n\rangle = \sum_{k=1}^{n}\langle \phi_1...\phi_{k-1}\ \delta\phi_k\ \phi_{k+1}...\phi_n\rangle$$

Let us focus on the case of conformal transformations. Using the fact that

$$\delta S[\phi] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ \partial^a v^b(\sigma)\ T_{ab}(\sigma)$$

where T_{ab} is the appropriate stress tensor for the action $S[\phi]$, gives,

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ (\partial^a v^b)\langle T_{ab}\ \phi_1...\phi_n\rangle = \sum_{k=1}^{n}\langle \phi_1...\phi_{k-1}\ \delta\phi_k\ \phi_{k+1}...\phi_n\rangle$$

remembering that $v^a(z)$ is not operator valued.

What is Σ here? Σ is the worldsheet with small discs removed around points where we insert operators. We shall denote the disc centred on z_i by $\mathscr{D}_i$ and the contour defining the boundary of the disc as $\partial\mathscr{D}_i = C_i$. We take curvee C_ω to encircle the point $\omega = z_k$ and not any other points. We take $v(z)$ to be zero in, and on the boundary of $\mathscr{D}_i$, when $i \neq k$ and a conformal vector inside and on C_ω.

$$v^a(z,\bar{z})|_{C_i} = 0, \qquad i \neq k$$

$$v^z(z,\bar{z})|_{C_\omega} = v(z) \qquad \text{and} \qquad v^{\bar{z}}(z,\bar{z})|_{C_\omega} = \bar{v}(\bar{z}).$$

We define Σ such that $\partial\Sigma = C_\omega \cup_{i\neq k} C_i$, so $v(z)$ is arbitrary on Σ (excluding the boundaries).

Thus $\delta\phi_k = 0$ unless $z_k = \omega$, giving

$$\langle \phi_1...\delta\phi(\omega,\bar{\omega})...\phi_n\rangle = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ \partial^a v^b(\sigma)\langle T_{ab}(\sigma)\ \phi_1...\phi_n\rangle$$

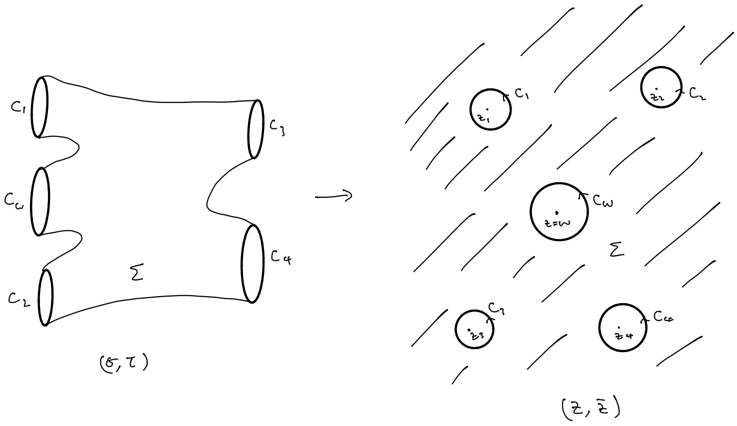


Figure 10.

Integrating by parts

$$\langle\phi_1...\delta_v\phi(\omega,\bar\omega)...\phi_n\rangle = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ \partial^a\Big(v^b(\sigma)\langle T_{ab}(\sigma)\phi_1...\phi_n\rangle\Big) - \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ v^b(\sigma)\partial^a\langle T_{ab}(\sigma)\phi_1...\phi_n\rangle$$

The first term on the RHS may be evaluated on $\partial\Sigma$. It is useful to use complex coordinates[24]

$$\begin{aligned}
&\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ \partial^a\Big(v^b(\sigma)\langle T_{ab}(\sigma)\phi_1...\phi_n\rangle\Big)\\
&= \frac{1}{2\pi i}\oint_{C_\omega} \mathrm{d}z\ v^z(z,\bar z)\langle T_{zz}(z,\bar z)\ \phi_1...\phi_n\rangle - \frac{1}{2\pi i}\oint_{C_\omega} \mathrm{d}\bar z\ v^{\bar z}(z,\bar z)\langle \bar T_{\bar z\bar z}(z,\bar z)\ \phi_1...\phi_n\rangle\\
&= \frac{1}{2\pi i}\oint_{C_\omega} \mathrm{d}z\ v(z)\ \langle T(z)\ \phi_1...\phi_n\rangle - \frac{1}{2\pi i}\oint_{C_\omega} \mathrm{d}\bar z\ \bar v(\bar z)\langle \bar T(\bar z)\phi_1...\phi_n\rangle
\end{aligned}$$

where we have use the fact that $v^a(z)$ vanishes on $C_{i\neq k}$ and is conformal on C_ω, i.e. $v^z(z,\bar z) = v(z)$ and $v^{\bar z}(z,\bar z) = \bar v(\bar z)$. We then have

$$\begin{aligned}
\langle\phi_1...\delta_v\phi(\omega,\bar\omega)...\phi_n\rangle = &\frac{1}{2\pi i}\oint_{C_1} \mathrm{d}z\ v(z)\ \langle T(z)\ \phi_1...\phi_n\rangle - \frac{1}{2\pi i}\oint_{C_1} \mathrm{d}\bar z\ \bar v\ (\bar z)\langle \bar T(\bar z)\ \phi_1...\phi_n\rangle\\
&-\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ v^a(z)\ \partial^b\langle T_{ab}(z,\bar z)\ \phi_1...\phi_n\rangle
\end{aligned}$$

The RHS includes an arbitrary function $v^a(z)$, whereas the LHS contains this function evaluated at ω, a point that is not included in Σ. We conclude that

$$\partial^a\langle T_{ab}(z,\bar z)\phi_1...\phi_n\rangle = 0,$$

[24] e.g.

$$\int_\Sigma \mathrm{d}z\wedge\mathrm{d}\bar z\ \partial_a j^a = \int_\Sigma \mathrm{d}z\wedge\mathrm{d}\bar z\ (\partial_z j^z + \partial_{\bar z} j^{\bar z}) = -\oint_{\partial\Sigma}\mathrm{d}\bar z\ j^z + \oint_{\partial\Sigma}\mathrm{d}z\ j^{\bar z} = -\oint_{\partial\Sigma}\mathrm{d}\bar z\ j_{\bar z} + \oint_{\partial\Sigma}\mathrm{d}z\ j_z$$

and so

$$\langle\phi_1...\delta_v\phi(\omega,\bar{\omega})...\phi_n\rangle = \frac{1}{2\pi i}\oint_{C_\omega} \mathrm{d}z\; v(z)\; \langle T(z)\; \phi_1...\phi_n\rangle - \frac{1}{2\pi i}\oint_{C_\omega} \mathrm{d}\bar{z}\; \bar{v}\;(\bar{z})\langle\bar{T}(\bar{z})\; \phi_1...\phi_n\rangle$$

We may therefore conclude that the abstract operator expression follows

$$\boxed{\delta_v\phi(\omega,\bar{\omega}) = \oint_{C_\omega}\frac{\mathrm{d}z}{2\pi i}\, v(z)\; T(z)\; \phi(\omega,\bar{\omega}) - \oint_{C_\omega}\frac{\mathrm{d}\bar{z}}{2\pi i}\,\bar{v}\;(\bar{z})\bar{T}(\bar{z})\;\phi(\omega,\bar{\omega})}$$

where this is taken to hold when inserted into a correlation function. The contour C_ω is taken to be any contour surrounding the point ω in the z-plane.

We see that the non-trivial contributions come from where

$$\lim_{z\to\omega} T(z)\; \phi(\omega,\bar{\omega}),$$

is singular. i.e. where there are poles in $T(z)\;\phi(\omega,\bar{\omega})$. This is related to the fact that, if these are operator expressions, we need to be careful about the ordering of the operators.

10.4 Radial Ordering and Symmetry Transformations

If we want to work with operator expressions, we need to think carefully about the ordering of the terms in correlation functions. Taking $z = e^{\tau+i\sigma}$, time ordering on the worldsheet becomes radial ordering: $\tau_1 > \tau_2 \equiv |z_1| > |z_2|$. We define *radial ordering* as

$$\boxed{\mathcal{R}\Big(A(z)B(\omega)\Big) := \begin{cases} A(z)B(\omega),\; |z| > |\omega| \\ B(\omega)A(z),\; |z| < |\omega| \end{cases}} \tag{10.1}$$

Consider the expression

$$\oint_{C(\omega)} \mathrm{d}z\mathcal{R}\Big(a(z)b(\omega)\Big)$$

where $C(\omega)$ is a contour on the z-plane around the point $z = \omega$. How do we radial order this expression? On some parts of the contour $|z| > |\omega|$, whilst on others $|z| < |\omega|$.We can use a simple trick of writing the contour as the difference of two contours, as shown in figure ??.

In the first contour we have that $|z| > |\omega|$ on all points of C_1, whilst $|z| < |\omega|$ on all points of C_2, thus this decomposition of the contour $C(\omega)$ is suitable for radial ordering expressions. We may therefore write

$$\oint_{C(\omega)} \mathrm{d}z\mathcal{R}\Big(a(z)b(\omega)\Big) = \oint_{C_1} \mathrm{d}z\mathcal{R}\Big(a(z)b(\omega)\Big) - \oint_{C_2} \mathrm{d}z\mathcal{R}\Big(a(z)b(\omega)\Big)$$

Since $|z| > |\omega|$ on C_1, we have

$$\oint_{C_1} \mathrm{d}z\; \mathcal{R}\Big(a(z)b(\omega)\Big) = \oint_{C_1} \mathrm{d}z\; a(z)b(\omega)$$

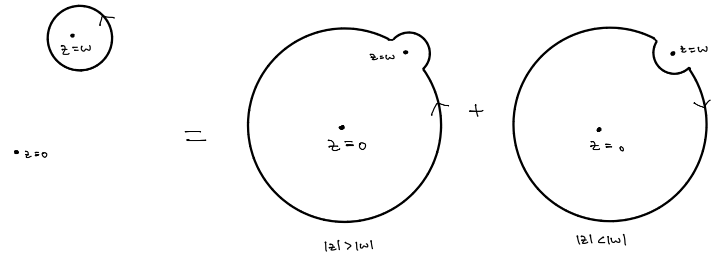


Figure 11. Making sense of radial ordering.

and since $|z| < |\omega|$ on C_2, we have

$$\oint_{C_2} \mathrm{d}z \, \mathcal{R}\Big(a(z)b(\omega)\Big) = \oint_{C_2} \mathrm{d}z \, b(\omega)a(z),$$

thus we have

$$\boxed{\oint_{C(\omega)} \mathrm{d}z \, \mathcal{R}\Big(a(z)b(\omega)\Big) = \oint_{C_1} \mathrm{d}z \, a(z)b(\omega) - \oint_{C_2} \mathrm{d}z \, b(\omega)a(z)}$$

We can define the operators

$$A = \oint_{C_1} \mathrm{d}z \, a(z), \qquad B = \oint_{C(0)} \mathrm{d}\omega \, b(\omega),$$

and if we now consider

$$\oint_{C(0)} \mathrm{d}\omega \oint_{C(\omega)} \mathrm{d}z \mathcal{R}\Big(a(z)b(\omega)\Big)$$

where $C(0)$ is a contour in the ω-plane surrounding the origin, we can write this as

$$\oint_{C(0)} \mathrm{d}\omega \oint_{|z|>|\omega|} \mathrm{d}z \, a(z)b(\omega) - \oint_{C(0)} \mathrm{d}\omega \oint_{|z|<|\omega|} \mathrm{d}z \, b(\omega)a(z) \equiv [A, B]$$

thus we can make sense of the commutator as

$$\boxed{\oint_{C(0)} \mathrm{d}\omega \oint_{C(\omega)} \mathrm{d}z \, \mathcal{R}\Big(a(z)b(\omega)\Big) = [A, B]}$$

And so we can make sense of our previous expression for the symmetry transformation (we focus on the holomorphic transformation here)

$$\begin{aligned}
\delta_\varepsilon \Phi(z) &= \oint_{C(z)} \frac{\mathrm{d}\omega}{2\pi i} \, \mathcal{R}\Big(v(\omega)T(\omega)\Phi(z)\Big) \\
&= \oint_{|\omega|>|z|} \frac{\mathrm{d}\omega}{2\pi i} \, v(\omega)T(\omega)\Phi(z) - \oint_{|\omega|<|z|} \frac{\mathrm{d}\omega}{2\pi i} \, \Phi(z)v(\omega)T(\omega)
\end{aligned}$$

and so we can make sense of the classical analogue of the statement (**??**) and state

$$\begin{aligned}\delta_v\phi(\omega,\bar\omega) &= \oint_{C_\omega}\frac{\mathrm{d}z}{2\pi i}\,\mathcal{R}\Big(v(z)\,T(z)\,\phi(\omega,\bar\omega)\Big) - \oint_{C_\omega}\frac{\mathrm{d}\bar z}{2\pi i}\,\mathcal{R}\Big(\bar v(\bar z)\,\bar T(\bar z)\,\phi(\omega,\bar\omega)\Big)\\ &= [Q,\phi(\omega,\bar\omega)]\end{aligned}$$

where

$$Q = \oint_{C_\omega}\frac{\mathrm{d}z}{2\pi i}\,v(z)\,T(z) - \oint_{C_\omega}\frac{\mathrm{d}\bar z}{2\pi i}\,\bar v(\bar z)\,\bar T(\bar z)$$

This is defined at fixed radius, which is the analogue of defining the Noether charge at fixed time.

11 Mode Expansions

If we put a Minkowski metric on the cylinder, we found that the equation of motion for the X^μ was naturally written in terms of the world-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, so that

$$X^\mu(\sigma,\tau) = X^\mu(\sigma^-) + \bar X^\mu(\sigma^-).$$

and we would then expand

$$X^\mu(\sigma^-) = x^\mu + p^\mu\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\Big(\alpha_n^\mu e^{-in\sigma^-} + \bar\alpha_n^\mu e^{-in\sigma^+}\Big).$$

A more natural object was

$$\partial_- X^\mu = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-n\sigma^-}.$$

We could study the same object on a worldsheet with Euclidean metric on the cylinder. The natural split is in terms of a complex coordinate $\omega = \tau + i\sigma$ and X^μ, which now obeys $\partial_\omega\partial_{\bar\omega}X^\mu = 0$, splits into holomorphic and anti-holomorphic parts [FACTOR OF i?]

$$\partial_\omega X^\mu = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n e^{-n\omega},$$

What does this look like on the conformal plane?

Imagine the general chiral primary field of weight h on the cylinder

$$\phi_{\text{cyl}}(\omega) = \sum_n \phi_n e^{-n\omega}$$

Transforming to the complex plane $z = e^\omega$ is a conformal transformation

$$\phi(z) = \left(\frac{\partial z}{\partial\omega}\right)^{-h}\phi_{\text{cyl}}(\omega) = z^{-h}\sum_n \phi_n e^{-n\omega},$$

and so the natural expansion of a chiral primary on the plane is

$$\boxed{\Phi(z) = \sum_n \phi_n z^{-n-h}}$$

This is the general form of the mode expansion of a chiral primary of weight h. More generally

$$\boxed{\Phi(z,\bar{z}) = \sum_{m,n} \phi_{mn} z^{-m-h} \bar{z}^{-n-\bar{h}}}$$

For example, the stress tensor, of weight $h = +2$, it is useful to write the stress tensor in terms of the modes L_n, where

$$T(z) = \sum_n L_n z^{-n-2}, \qquad \bar{T} = \sum_n \bar{L}_n \bar{z}^{-n-2}.$$

For later reference we also note

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu z^{-n} + \bar{\alpha}_n^\mu \bar{z}^{-n} \right)$$

11.1 States and Operators

We have thus far focussed on the space of operators. These operators have a natural action on the Hilbert space of the theory $\mathcal{H}$. In fact, a special feature of two dimensional CFT is that the states of the theory are in one-to-one correspondence with the operators. We shall discuss this more later. Consider the mode expansion of the weight-one chiral field[25]

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu z^{-n-1}$$

The modes with $n > 0$ are annihilation operators and they define a vacuum $|0\rangle$ such that

$$\alpha_n^\mu |0\rangle = 0, \qquad n \geq 0.$$

and similarly

$$\langle 0| \alpha_n^\mu = 0, \qquad n \leq 0.$$

A state corresponding to the field $\Phi(z)$ is given by choosing local coordinates z and inserting the operator at $z = 0$, giving the relationship

$$\boxed{\lim_{z \to 0} \Phi(z)|0\rangle = |\Phi\rangle}$$

[25] One might think that, for pedagogical purposes it is better to use $X^\mu(z)$ and not its derivative; however, we will see later that $X^\mu(z)$ is not a conformal primary.

11.2 Relationship between normal and radial ordering

Consider the weight-one chiral field

$$\partial X^\mu(z) = \sum_n \alpha_n^\mu z^{-n-1} := j^\mu(z).$$

We split this into creation ($j_-^\mu(z)$) and annihilation ($j_+^\mu(z)$) operator components

$$j^\mu(z) = j_+^\mu(z) + j_-^\mu(z)$$

where

$$j_+^\mu(z) = \sum_{n>0} \alpha_n^\mu z^{-n-1}, \qquad j_-^\mu(z) = \sum_{n\geq 0} \alpha_{-n}^\mu z^{n-1}$$

where we have included the $n = 0$ term in $j_-^\mu(z)$. We define the *normal ordering* of an operator expression by putting all of the creation operators to the left of the annihilation operators, thus the expectation of any normal ordered product will vanish. For example:

$$: j^\mu(z) j^\nu(\omega) := j_+^\mu(z) j_+^\nu(\omega) + j_-^\mu(z) j_+^\nu(\omega) + j_-^\nu(\omega) j_+^\mu(z) + j_-^\mu(z) j_-^\nu(\omega)$$

Note that

$$: j^\mu(z) j^\nu(\omega) := j^\mu(z) j^\nu(\omega) + [j_-^\nu(\omega), j_+^\mu(z)]$$

Let us use the mode expansion to evaluate this commutator

$$\begin{aligned}
[j_-^\nu(\omega), j_+^\mu(z)] &= \sum_{m\geq 0, n>0} [\alpha_{-m}^\nu, \alpha_n^\mu] \omega^{m-1} z^{-n-1} \\
&= -\sum_{m,n>0} n \delta_{mn} \eta^{\mu\nu} \omega^{m-1} z^{-n-1} \\
&= -\frac{\eta^{\mu\nu}}{z^2} \sum_{n>0} n \left(\frac{\omega}{z}\right)^{n-1} \\
&= \frac{\eta^{\mu\nu}}{(z-\omega)^2}
\end{aligned}$$

The series converges if we assume $|z| > |\omega|$. And so we have

$$j^\mu(z) j^\nu(\omega) - : j^\mu(z) j^\nu(\omega) := \frac{\eta^{\mu\nu}}{(z-\omega)^2}, \qquad \text{if } |z| > |\omega|.$$

If we swap $z \leftrightarrow \omega$ and $\mu \leftrightarrow \nu$ in the above expression we find

$$j^\nu(\omega) j^\mu(z) - : j^\mu(z) j^\nu(\omega) := \frac{\eta^{\mu\nu}}{(z-\omega)^2}, \qquad \text{if } |z| < |\omega|.$$

The two expressions may be combined to give

$$\boxed{\mathcal{R}\Big(j^\mu(z) j^\nu(\omega)\Big) - : j^\mu(z) j^\nu(\omega) := \frac{\eta^{\mu\nu}}{(z-\omega)^2}}$$

This is a simple example of Wick's theorem, which you should have already come across before in Quantum Field Theory. We often denote the contraction by

$$\wick{\c j^{\nu}(\omega)\c j^{\mu}(z)} = \frac{\eta^{\mu\nu}}{(z-\omega)^2}.$$

In general

$$\mathcal{R}\Big(\Phi_1(z_1)...\Phi_n(z_n)\Big) = :\Phi_1(z_1)...\Phi_n(z_n): + \sum_{(i,j)} :\Phi_1(z_1)...\wick{\c\Phi_i(z_i)...\c\Phi_j(z_j)}...\Phi_n(z_n):$$
$$+\sum_{(i,j),(k,l)} :\Phi_1(z_1)...\wick{\c1\Phi_i(z_i)...\c2\Phi_k(z_j)...\c1\Phi_j(z_i)...\c2\Phi_l(z_j)}...\Phi_n(z_n): + ...$$

where the sums are taken over all distinct pairs. For example

$$\begin{aligned}\mathcal{R}\Big(\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4)\Big) &= :\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4): \\ &+\Big(:\Phi_1(z_1)\Phi_2(z_2):\wick{\c\Phi_3(z_3)\c\Phi_4(z_4)} + 2\text{ terms}\Big) \\ &+\Big(\wick{\c\Phi_1(z_1)\c\Phi_2(z_2)}\,\wick{\c\Phi_3(z_3)\c\Phi_4(z_4)} + 2\text{ terms}\Big)\end{aligned}$$

and so

$$\Big\langle\mathcal{R}\Big(\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4)\Big)\Big\rangle = \wick{\c\Phi_1(z_1)\c\Phi_2(z_2)}\,\wick{\c\Phi_3(z_3)\c\Phi_4(z_4)} + 2\text{ terms}.$$

and since

$$\Big\langle\mathcal{R}\Big(\Phi_1(z_1)\Phi_2(z_2)\Big)\Big\rangle = \wick{\c\Phi_1(z_1)\c\Phi_2(z_2)},$$

we have for a free theory

$$\Big\langle\mathcal{R}\Big(\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4)\Big)\Big\rangle = \Big\langle\mathcal{R}\Big(\Phi_1(z_1)\Phi_2(z_2)\Big)\Big\rangle\Big\langle\mathcal{R}\Big(\Phi_3(z_3)\Phi_4(z_4)\Big)\Big\rangle + 2\text{ terms}.$$

It will be understood that correlation functions are radially ordered unless stated otherwise and we will not always explicitly include the radial ordering explicitly. Compactly the above result gives

$$\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle = \langle\Phi_1\Phi_2\rangle\langle\Phi_3\Phi_4\rangle + \langle\Phi_1\Phi_3\rangle\langle\Phi_2\Phi_4\rangle + \langle\Phi_1\Phi_4\rangle\langle\Phi_2\Phi_3\rangle.$$

Wick's theorem gives a way of removing divergences from a radially ordered string of operators[26].

[26] In the path integral formalism we have classical functions, not operators; however, a careful treatment of the time-slicing construction of the path integral leads to a natural time ordering when path integral expressions are defined carefully. The need for such care is obvious when we must be able to derive results such as $\langle[X,P]\rangle = i\hbar$ from the path integral where X and P are both classical functions (but defined on different time slices).

12 Operator Product Expansions

Operator product expansions are a useful idea in many areas of quantum field theory, reaching from QCD to our present concern in Strung Theory. The basic idea is to describe a theory in terms of the short range behaviour of its operators. For example, if we have a theory with local operators $\mathcal{O}_i(z)$, then the operator product expansion tells us what happens when we move two such operators close together

$$\lim_{\omega\to z} \mathcal{O}_i(\omega)\mathcal{O}_j(z) = \sum_k f_{ij}{}^k(z-\omega)\mathcal{O}_k(z).$$

In general there will be singularities as $\omega \to z$. This is not a problem, in fact for our purposes, it is the interesting part of the OPE. Many of our expressions involve contour integrals and it will be the pole structure of the OPE which contains the informations we are after in most cases.

12.1 $X^\mu(z)X^\nu(\omega)$ OPE

We will see that the two-point correlation function for a free theory is simply the Green's function for the $X^\mu(z,\bar{z})$ equation of motion

$$\frac{1}{4\pi\alpha'}\partial\bar{\partial}X^\mu(z,\bar{z}) = 0.$$

We are therefore looking for a function $G(z\omega)$ that satisfies

$$\partial\bar{\partial}G(z\omega) = 2\pi\delta^2(z-\omega).$$

Using the result

$$\bar{\partial}\left(\frac{1}{z-\omega}\right) = 2\pi\delta^2(z-\omega),$$

we see that

$$\partial\bar{\partial}G(z\omega) = \frac{1}{z-\omega},$$

and so

$$G(z,\bar{z};\omega,\bar{\omega}) = \ln|z-\omega|^2.$$

The general solution to the equation of motion is the sum of holomorphic and anti-holomorphic functions

$$X^\mu(z,\bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z}).$$

and the correlation functions are

$$\langle X^\mu(z)X^\nu(\omega)\rangle = \eta^{\mu\nu}\ln(z-\omega), \qquad \langle \bar{X}^\mu(\bar{z})\bar{X}^\nu(\bar{\omega})\rangle = \eta^{\mu\nu}\ln(\bar{z}-\bar{\omega})$$

and the OPEs are

$$X^\mu(z)X^\nu(\omega) = \eta^{\mu\nu}\ln(z-\omega) + ..., \qquad \bar{X}^\mu(\bar{z})\bar{X}^\nu(\bar{\omega}) = \eta^{\mu\nu}\ln(\bar{z}-\bar{\omega}) + ...$$

whereas $X^\mu(z)\bar{X}^\nu(\bar{\omega})$ is regular.

12.1.1 Composite Operators

Note that

$$\overbrace{X^\mu(z)X}^{}{}^\nu(\omega) = -\frac{\alpha'}{2}\ln(z-\omega)$$

and so

$$\overbrace{\partial X^\mu(z)\partial X}^{}{}^\nu(\omega) = -\frac{\alpha'}{2}\frac{1}{(z-\omega)^2}$$

We can then see that the correct definition of the stress tensor is

$$\boxed{T(z) = -\frac{1}{\alpha'}\lim_{\omega\to z}\left(\partial X^\mu(z)\partial X^\nu(\omega) + \frac{\alpha'}{2}\frac{1}{(z-\omega)^2}\right)}$$

Thus the OPEs allow us to correctly define composite operators in a meaningful way.

12.2 $T(z)X^\mu(\omega)$ OPE and conformal transformations

We shall implicitly assume radial ordering in what follows. We shall be interested in the OPE of the stress tensor with functions of X.

12.2.1 $T(z)X^\mu(\omega)$ OPE

We are interested in

$$T(z)X^\mu(\omega) = -\frac{1}{\alpha'} : \partial X^\nu \partial X_\nu(z) : X^\mu(\omega)$$

Differentiating with respect to z and ω, we find the expressions

$$\partial X^\mu(z)X^\nu(\omega) = -\frac{\alpha'}{2}\frac{1}{z-\omega} + ...$$

and so

$$\begin{aligned}
T(z)X^\mu(\omega) &= -\frac{1}{\alpha'} : \partial X^\nu \partial X_\nu(z) : X^\mu(\omega) \\
&= -\frac{2}{\alpha'} : \partial X^\nu \overbrace{\partial X_\nu(z) : X}^{}{}^\mu(\omega) + ... \\
&= -\frac{2}{\alpha'}\partial X_\nu(z)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-\omega}\right) + ... \\
&= \frac{\partial X^\mu(z)}{z-\omega} + ...
\end{aligned}$$

We may then expand $\partial X^\mu(z)$ around ω

$$\partial X^\mu(z) = \partial X^\mu(\omega) + (z-\omega)\partial^2 X^\mu(\omega) + ...$$

The divergent term is then

$$\boxed{T(z)X^\mu(\omega) = \frac{1}{z-\omega}\partial X^\mu(\omega) + ...}$$

It is not hard to show that this gives the correct conformal transformation for $X^\mu(z)$. In fact we can use the known conformal transformations of fields to determine the OPE with the stress tensor. For a chiral primary field of weight h

$$\delta_v \Phi(z) = \oint_{C(z)} \frac{\mathrm{d}\omega}{2\pi i}\, \mathcal{R}\Big(T(\omega)v(\omega)\Phi(z)\Big)$$

But we know that

$$\delta_v \Phi(z) = v(z)\, \partial\Phi(z) + h\partial v(z)\, \Phi(z),$$

so, using the residue theorem,

$$\boxed{\frac{1}{(n-1)!}\partial_z^{n-1} f(z) = \frac{1}{2\pi i}\oint_{C(z)} \mathrm{d}\omega \frac{f(\omega)}{(\omega - z)^n}}$$

we have that

$$\oint_{C(z)} \frac{\mathrm{d}\omega}{2\pi i}\mathcal{R}\Big(T(\omega)v(\omega)\Phi(z)\Big) = \oint_{C(z)} \frac{\mathrm{d}\omega}{2\pi i} v(\omega)\left(\frac{h}{(z-\omega)^2}\Phi(z) + \frac{1}{z-\omega}\partial\Phi(z) + \text{ Regular terms}\right)$$

and so the OPE is

$$\boxed{T(\omega)\Phi(z) = \frac{h}{(z-\omega)^2}\Phi(z) + \frac{1}{z-\omega}\partial\Phi(z) + ...}$$

12.2.2 $T(z)\, e^{ik\cdot X(\omega)}$ OPE

$X^\mu(z,\bar{z})$ is not a conformal primary; however $\partial X^\mu(z,\bar{z})$ is. We can trace this fact back to the XX OPE being a logarithm, whereas derivatives of X have OPEs are negative powers of the difference for the arguments. One might imagine that exponentials of X also stands a chance of having OPEs of the form **??**.

$$T(z) : e^{ik\cdot X(\omega)} := T(z) \sum_{n\geq 0} \frac{(i)^n}{n!} k_{\mu_1}...k_{\mu_n} : X^{\mu_1}(\omega)...X^{\mu_n}(\omega) :$$

Single contractions contribute

$$\begin{aligned}
&-\frac{2}{\alpha'}\,\partial X\cdot\partial X(z) : \sum_{n\geq 0} \frac{(i)^n}{n!} k_{\mu_1}...k_\nu...k_{\mu_n} : X^{\mu_1}(\omega)...X^\nu(\omega)...X^{\mu_n}(\omega) \\
&= \sum_{n>0} \frac{(i)^n}{n!} n\Big(k\cdot X(\omega)\Big)^{n-1} k_\nu \left(\frac{1}{z-\omega}\partial X^\nu(\omega)\right) \\
&= \sum_{m\geq 0} \frac{(i)^m}{m!}\Big(k\cdot X(\omega)\Big)^{m}\, ik_\nu \left(\frac{1}{z-\omega}\partial X^\nu(\omega)\right) \\
&= \frac{1}{z-\omega}\partial\Big(e^{ik\cdot X(\omega)}\Big),
\end{aligned}$$

whilst double contractions contribute

$$
\begin{aligned}
&-\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu(z) : \sum_{(i,j)} \sum_{n\geq 0} \frac{(i)^n}{n!} k_{\mu_1} ... k_{\mu_i} ... k_{\mu_j} ... k_{\mu_n} X^{\mu_1}(\omega) ... X^{\mu_i}(\omega) ... X^{\mu_j}(\omega) ... X^{\mu_n}(\omega) : \\
&= -\frac{1}{\alpha'} \sum_{n\geq 2} k_{\mu_2} ... k_{\mu_n} \frac{i^n}{n!} n(n-1)\, X^{\mu_2} ... X^{\mu_n} \left(-\frac{\alpha'}{2}\right)^2 \frac{k^2}{(z-\omega)^2} \\
&= -\frac{\alpha'}{4} \frac{k^2}{(z-\omega)^2} \sum_{n\geq 2} \Big(k\cdot X(\omega)\Big)^{n-2} i^2 i^{n-2} \frac{n!}{n!(n-2)!} \\
&= \frac{\alpha'}{4} \frac{k^2}{(z-\omega)^2} e^{ik\cdot X(\omega)}, \qquad (12.1)
\end{aligned}
$$

so we have

$$
\boxed{T(z) : e^{ik\cdot X(\omega)} := \left(\frac{\alpha' k^2/4}{(z-\omega)^2} + \frac{\partial}{z-\omega} \right) : e^{ik\cdot X(\omega)} : + ...}
$$

And so we deduce that the conformal weight of the operator $: e^{ik\cdot X(z,\bar{z})}$ is

$$
(h, \bar{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4} \right).
$$

This fact will become important when we consider momentum space wave-functions of the physical states.

12.3 $T(z)T(\omega)$ OPE and the Virasoro Algebra

We saw that the $T\Phi$ OPE described how the field Φ transforms under conformal transformations. We now consider how the stress tensor transforms under conformal transformations by calculating the TT OPE.

12.3.1 The $T(z)T(\omega)$ OPE

We have that

$$
T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu(z) :, \qquad \partial X^\mu(z) \partial X^\nu(\omega) = -\frac{\alpha'}{2} \frac{1}{(z-\omega)^2}.
$$

We are then interested in

$$
\begin{aligned}
T(z)T(\omega) = 4 \times \left(-\frac{1}{\alpha'} :\right)^2 &: \partial X^\mu(z) \partial X_\mu(z) \partial X^\nu(\omega) \partial X_\nu(\omega) : \\
+2 \times \left(-\frac{1}{\alpha'} :\right)^2 &: \partial X^\mu(z) \partial X_\mu(z) \partial X^\nu(\omega) \partial X_\nu(\omega) : + ...
\end{aligned}
$$

where the first term includes the four possible double contractions (each giving the same result) and the second the two possible double contractions. We have then

$$
T(z)T(\omega) = -\frac{2}{\alpha'} \frac{\eta_{\mu\nu}}{(z-\omega)^2} : \partial X^\mu(z) \partial X^\nu(\omega) : + \frac{1}{2} \frac{\delta^\mu_\nu}{(z-\omega)^2} \frac{\delta^\nu_\mu}{(z-\omega)^2} + ...
$$

Notice that $:\partial X^\mu(z)\partial X^\nu(\omega):$ is not proportional to the stress tensor as it is bi-local. If we expand $\partial X^\mu(z)$ about $z=\omega$,

$$\partial X^\mu(z) = \partial X^\mu(\omega) + (z-\omega)\partial^2 X^\mu(\omega) + ...$$

and so

$$T(z)T(\omega) = \frac{D/2}{(z-\omega)^4} - \frac{2}{\alpha'}\frac{1}{(z-\omega)^2} : \partial X_\mu \partial X^\mu(\omega) : - \frac{1}{\alpha'}\frac{2}{z-\omega} : \partial X_\mu \partial^2 X^\mu(\omega) : + ...$$

Using the fact that

$$\partial T(\omega) = -\frac{2}{\alpha'} : \partial^2 X \cdot \partial X(\omega) :,$$

we have

$$\boxed{T(z)T(\omega) = \frac{D/2}{(z-\omega)^4} + \frac{2}{(z-\omega)^2}T(\omega) + \frac{1}{z-\omega}\partial T(\omega) + ...}$$

where we implicitly assume radial ordering, so $|z| \geq |\omega|$.

12.3.2 The Virasoro Algebra

For example, the stress tensor, of weight $h=+2$, it is useful to write the stress tensor in terms of the modes L_n, where

$$T(z) = \sum_n L_n z^{-n-2}, \qquad \bar{T} = \sum_n \bar{L}_n \bar{z}^{-n-2}.$$

which may be inverted to give

$$L_m = \frac{1}{2\pi i}\oint_C \mathrm{d}z z^{m+1}\, T(z)$$

The commutator of two such generators is

$$[L_m, L_n] = \oint_{C_z(0)} \frac{\mathrm{d}z}{2\pi i} \oint_{C_\omega(0)} \frac{\mathrm{d}\omega}{2\pi i} z^{m+1}\omega^{n+1}[T(z), T(\omega)]$$

What do we mean by this commutator? Recall, the definition based on radial ordering

$$\begin{aligned} \oint_{C_z(0)} \mathrm{d}z\; z^{m+1}[T(z), T(\omega)] &:= \oint_{|z|>|\omega|} \mathrm{d}z\; T(z)T(\omega) - \oint_{|z|<|\omega|} \mathrm{d}z\; T(\omega)T(z) \\ &= \oint_{C(\omega)} \mathrm{d}z\; z^{m+1}\mathcal{R}\Big(T(z)T(\omega)\Big) \end{aligned} \tag{12.2}$$

so we have

$$[L_m, L_n] = \oint_{C(0)} \frac{\mathrm{d}\omega}{2\pi i}\omega^{n+1} \oint_{C(\omega)} \frac{\mathrm{d}z}{2\pi i} z^{m+1}\, \mathcal{R}\Big(T(z)T(\omega)\Big)$$

Using the TT OPE we have

$$\begin{aligned}[L_m, L_n] &= \oint_{C(0)} \frac{\mathrm{d}\omega}{2\pi i}\omega^{n+1} \oint_{C(\omega)} \frac{\mathrm{d}z}{2\pi i} z^{m+1} \left(\frac{D/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} \right) \\ &= \frac{D}{12}(m^3 - m)\delta_{m,-n} + (m-n)L_{m+n}. \end{aligned} \tag{12.3}$$

and so we see that the Wit algebra acquires a central extension in the quantum theory

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m,-n}}$$

We call this algebra the Virasoro algebra. It is not hard to see that $\bar{L}_m$ will give another copy of the algebra and $[L_m, \bar{L}_n] = 0$.

13 The b, c Ghost System

We now bring all of our experience thus far to study this important CFT for string theory. The action may be written as

$$S[b,c] = \frac{i}{2\pi} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{ac} b_{ab} \nabla_c c^b$$

$b_{ab}(z, \bar{z})$ is a symmetric traceless tensor field and $c^a(z, \bar{z}$ is a vector field. The stress tensor is given by varying the action with respect to the metric and we see that the ghost terms give a contribution to the metric that we have not yet considered. The effect of the ghosts is to give the contribution to the total stress tensor (remember there is metric-dependence in the connection ∇_a as well as the obvious $\sqrt{-h}h^{ac}$ factor)

$$T^{\mathrm{gh}}_{ab} = -i\left(\frac{1}{2}c^c\nabla_{(a}b_{b)c} + (\nabla_{(a}c^c)b_{b)c} - \mathrm{trace}\right).$$

We to work with a Euclidean metric. In conformal gauge $h_{ab} = e^\phi \delta_{ab}$ with $(z, \bar{z})$ coordinates, the ghost action becomes

$$S[b,c] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2 z\, b_{zz}\partial_{\bar{z}} c^z + \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2 z\, b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}$$

The two degrees of freedom in c^a may then be written as

$$c^z = c, \qquad c^{\bar{z}} = \bar{c}$$

b_{ab} is symmetric and traceless and so has two real degrees of freedom which, in these coordinates may be written as

$$b_{zz} = b, \qquad b_{\bar{z}\bar{z}} = \bar{b}.$$

The total action may be written as

$$\boxed{S[X,b,c] = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2z\ \eta_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu + \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2z\ b\bar{\partial}c + \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2z\ \bar{b}\partial\bar{c}}$$

These holomorphic and anti-holomorphic sectors clearly do not interact (the Hilbert space factorizes) so it is sufficient to consider just the holomorphic sector. A new feature for us is the fact that these ghosts do not obey the spin-statistics theorem[27] and are bosonic fields with Fermi statistics. The matter and ghost stress tensors may be incorporated into a total stress tensor

$$T(z) = T_X(z) + T_{\mathrm{gh}}(z),$$

where $T_X(z)$ is the contribution to the stress tensor coming from the embedding fields $X^\mu(z)$. Explicitly, we have

$$\boxed{T_X(z) = -\frac{1}{\alpha'} :\partial X^\mu\partial X_\mu(z):, \qquad T_{\mathrm{gh}}(z) =: (\partial b)c(z) : -2\partial(:bc(z):).}$$

The b and c ghosts are clearly canonically conjugate. The action gives the canonical anti-commutation relations by the usual route of replacing the Poisson brackets of the classical theory with equal time commutators (or anti-commutators, in this case)

$$\{b(z), c(\omega)\}$$

where

$$\{A, B\} := AB + BA$$

for Grassmann quantities A and B. We shall instead derive such anti-commutators for the modes from the OPE directly and not rely on Poisson brackets.

13.1 OPEs

The ghost system is a free theory and so we may use Wick's theorem, which states that

$$\mathcal{R}\Big(b(z)c(\omega)\Big) =: b(z)c(\omega) : + \overbrace{b(z)c}(\omega)\,.$$

Taking expectations of both sides gives the contraction as the two-point function

$$\Big\langle\mathcal{R}\Big(b(z)c(\omega)\Big)\Big\rangle = \overbrace{b(z)c}(\omega)\,.$$

Our first task then is to compute the two-point function for the ghost system.

The tree-level two point function is simply the classical Green's function. Since the theory is free, this is the whole story. To find the Green's function $S(z,\omega)$, which must satisfy

$$\frac{1}{2\pi}\bar{\partial}S(z,\omega) = \delta^2(z-\omega)$$

[27] Once has to be a little careful about what one means by spin in 2D.

we make use of the result on the sphere that[28]

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{z-\omega}\right) = 2\pi\delta^2(z-\omega),$$

[29] We therefore identify the Green's function for the ghosts as

$$S(z,\omega) = \frac{1}{z-\omega}.$$

We then have that

$$\langle b(z)c(\omega)\rangle = \contraction{}{b}{(z)}{c}b(z)c(\omega) = \frac{1}{z-\omega}$$

which means that the OPE is simply

$$\boxed{b(z)c(\omega) = \frac{1}{z-\omega} + \ldots}$$

and so the correct definition of the ghost stress tensor is[30]

$$\boxed{T_{\text{gh}}(z) =: \lim_{z\to\omega}\left(-2b(\omega)\,\partial c(z) + \partial b(\omega)\,c(z) + \frac{1}{(z-\omega)^2}\right)}$$

13.1.1 Conformal Transformations from OPEs

Now that we have the basic OPE of the ghost theory, we can calculate OPEs of composite operators. We introduce the Noether current $j_a = v^b T_{ab}$ and compute the conformal transformation generated by the stress tensor. We shall compute $\delta_v b(z)$ explicitly. The calculation of the $c(z)$ transformation follows straightforwardly.

$$\begin{aligned}
\mathcal{R}\Big(T(z)b(\omega)\Big) &= :\partial b(z)\,\contraction{}{c}{(z)}{b}c(z)b(\omega): - :2\partial(b\,\contraction{}{c}{(z))}{b}c(z))b(\omega): + \ldots \\
&= \frac{\partial b(z)}{z-\omega} - 2\partial_z\left(\frac{b(z)}{z-\omega}\right) + \ldots \\
&= \frac{2}{(z-\omega)^2}b(z) - \frac{1}{z-\omega}\partial b(z) + \ldots
\end{aligned}$$

Expanding around $z=\omega$

$$b(z) = b(\omega) + (z-\omega)\partial b(\omega) + \ldots$$

[28] One may check this by integrating over a region in the complex plane and then using Stoke's theorem.

[29] There are no globally holomorphic zero modes of $\bar{\partial}$ on the sphere. This changes at higher genus.

[30] Where the fact that

$$\partial_z \frac{1}{z-\omega} = -\frac{1}{(z-\omega)^2}$$

has been used.

gives

$$\boxed{T_{\text{gh}}(z)b(\omega) = \frac{2}{(z-\omega)^2}b(\omega) + \frac{1}{z-\omega}\partial b(\omega) + ...}$$

and so we find that $b(z)$ has weight $(h, \bar{h}) = (2, 0)$. A similar calculation gives

$$\boxed{T_{\text{gh}}(z)c(\omega) = \frac{-1}{(z-\omega)^2}c(\omega) + \frac{1}{z-\omega}\partial c(\omega) + ...}$$

so that $c(z)$ has weight $(h, \bar{h}) = (-1, 0)$. Both are primary fields.

We now consider the conformal transformation of the stress tensor. The $T_{gh}T_{gh}$ OPE is

$$\boxed{T_{gh}(z)T_{gh}(\omega) = \frac{-26/2}{(z-\omega)^4} + \frac{2}{(z-\omega)^2}T_{\text{gh}}(\omega) + \frac{1}{z-\omega}\partial T_{\text{gh}}(\omega) + ...}$$

13.2 Total Stress Tensor and the Critical Dimension

The total stress tensor is the sum of the matter and ghost stress tensor

$$\mathcal{T}(z) = T_X(z) + T_{\text{gh}}(z).$$

The OPE between the ghost and matter sectors are trivial $T_X(z)T_{\text{gh}}(\omega) = 0 + ...$ and so the OPE of the total stress tensor with itself is

$$\mathcal{T}(z)\mathcal{T}(\omega) = \frac{(D-26)}{2(z-\omega)^4} + \frac{2\mathcal{T}(\omega)}{(z-\omega)^2} + \frac{\partial\mathcal{T}(\omega)}{z-\omega} + ...$$

so that if

$$\boxed{D = 26}$$

then the total stress tensor has no anomaly and the theory is conformal at the quantum level. This is a remarkable result: the dimension of spacetime is fixed by the quantum consistency of the theory!

13.3 Mode expansions

In terms of mode expansions, we have

$$b(z) = \sum_n b_n z^{-n-2}, \qquad c(z) = \sum_n c_n z^{-n+1}.$$

The canonical anticommutation relations become

$$\boxed{\{b_m, c_n\} = \delta_{m,-n}}$$

Using this, it is not hard to check the validity of the contraction $\overbrace{b(z)c(\omega)}$ above . Perhaps a more direct (and satisfying?) route is to use the OPE to compute this commutator directly. We may write the modes as

$$b_n = \frac{1}{2\pi i}\oint_{C(0)} \mathrm{d}z\ z^{n+1}b(z), \qquad c_n = \frac{1}{2\pi i}\oint_{C(0)} \mathrm{d}z\ z^{n-2}c(z).$$

Then

$$
\begin{aligned}
\{b_m, c_n\} &= \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^{m+1} \oint_{\omega=0} \frac{\mathrm{d}\omega}{2\pi i} \omega^{n-2} \{b(z), c(\omega)\} \\
&= \oint_{\omega=0} \frac{\mathrm{d}\omega}{2\pi i} \omega^{n-2} \oint_{z=\omega} \frac{\mathrm{d}z}{2\pi i} z^{m+1} \mathcal{R}\Big(b(z)c(\omega)\Big)
\end{aligned}
$$

We can now use the OPE to evaluate the contour integral

$$
\begin{aligned}
\{b_m, c_n\} &= \oint_{\omega=0} \frac{\mathrm{d}\omega}{2\pi i} \omega^{n-2} \oint_{z=\omega} \frac{\mathrm{d}z}{2\pi i} z^{m+1} \left(\frac{1}{z-\omega} + \text{ Regular terms} \right) \\
&= \oint_{\omega=0} \frac{\mathrm{d}\omega}{2\pi i} \omega^{m+n-1} = \delta_{m,-n}.
\end{aligned}
$$

and so we recover the mode commutator without further recourse to the classical theory.

14 BRST Symmetry

In the path integral it is sometimes useful to keep the gauge-fixing part explicit and write

$$
\mathcal{Z} = \frac{1}{|\mathrm{CKG}|} \int_{\mathcal{M}_g} \mathrm{d}^s t \int \mathcal{D}X \mathcal{D}h \, \mathcal{D}c \, \mathcal{D}b \, e^{iS[X,\hat{h},b,c]} \, \delta[h - \hat{h}] \prod_{I=1}^{s} (b|\mu_I) \prod_{i,a} c^a(\hat{\sigma}_i)
$$

On the support of this delta-functional we can exchange h_{ab} and $\hat{h}_{ab}$. It will also be useful to incorporate the delta-functional into the action. To this end, we introduce the symmetric, traceless fields B^{ab} and introduce

$$
S_{\mathrm{gf}}[B, h] = \frac{1}{4\pi} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} B^{ab} \Big(\delta_{ab} - h_{ab} \Big),
$$

to the action. Functional integration over B^{ab} gives the gauge-fixing condition $h_{ab} = \delta_{ab}$. The full action is now

$$
\begin{aligned}
S[X, h, b, c, B] = &-\frac{1}{4\pi\alpha'} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{i}{2\pi} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} b_{ab} \nabla^a c^b \\
&+ \frac{1}{4\pi} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} B^{ab} \Big(\delta_{ab} - h_{ab} \Big) \qquad (14.1)
\end{aligned}
$$

The quantization of the ghosts follows from this action and the fact that the are Grassmann fields. The key point is that $b_{ab}(\tau, \sigma)$ and $c^a(\tau, \sigma)$ are conjugate fields. It is worth pointing out that the ghosts have integer spin but fermionic statistics. They are not physical observables and this violation of the spin-statistics theorem is the hallmark of ghost fields.

The BRST symmetry is a remnant of the original gauge symmetry. The idea is that the gauge-fixed theory has a residual rigid symmetry - the BRST symmetry, where we replace the gauge parameter $v(z)$ with $\epsilon\, c(z)$ where ϵ is a constant anti-commuting parameter and $c(z)$ is the weight -1 ghost field. The fields thus transform as

$$\begin{aligned}
\delta_Q X^\mu &= i\epsilon c^a \partial_a X^\mu \\
\delta_Q h_{ab} &= \epsilon(\mathcal{P}c)_{ab} \\
\delta_Q c^a &= i\epsilon c^b \partial_b c^a, \\
\delta_Q b_{ab} &= i\epsilon B_{ab}, \qquad \delta_Q B_{ab} = 0
\end{aligned}$$

where $\mathcal{P}$ is the operator whose kernel gives the conformal Killing vectors, defined in the previous set of notes. The original action

$$S[X,h] = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\, \sqrt{h} h^{ab}\partial_a X^\mu \partial_b X^\nu$$

is invariant under this transformation as the BRST transformation is simply a gauge transformation with (field-dependent) parameter $v^a = \epsilon\, c^a$.

To see that the remaining terms are BRST-invariant we define a *gauge-fixing fermion*

$$\Psi[b,h] = \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma \sqrt{h} b^{ab}\Big(\delta_{ab} - h_{ab}\Big),$$

which transforms as

$$\delta_Q \Psi = \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}B^{ab}\Big(\delta_{ab} - h_{ab}\Big) + \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}b^{ab}(\mathcal{P}c)_{ab}$$

This is precisely the ghost and gauge-fixing terms in the action. So we see that the full action

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\, \sqrt{h}h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}b^{ab}(\mathcal{P}c)_{ab} + \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}B^{ab}\Big(\delta_{ab} - h_{ab}\Big)$$

may be written as

$$S[X,h] + \delta_Q\Psi.$$

The last thing we need to note is that classically, the BRST transformation is nilpotent, i.e. $\delta_Q^2\Phi = 0$ on any field Φ, thus the full action is manifestly invariant under classical BRST transformations since $\delta_Q^2\Psi = 0$. The nilpotency of the BRST symmetry plays an important role in determining the physical spectrum of the theory and the requirement that $\delta_Q^2\Phi = 0$ holds at the quantum level will be of crucial importance to the quantum consistency of the theory.

The equation of motion for the metric is modified to

$$T_{ab} = B_{ab},$$

where this T_{ab} is the total ghost and matter stress tensor. If we now integrate out the B_{ab} Lagrange multiplier, the gauge choice $h_{ab} = \delta_{ab}$ is imposed and the effective action is

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\ \eta_{\mu\nu}\partial_a X^\mu \partial^a X^\nu + \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\ b^{ab}(\mathcal{P}c)_{ab}$$

with the effective BRST transformations

$$\begin{aligned}\delta_Q X^\mu &= i\epsilon c^a \partial_a X^\mu\\ \delta_Q c^a &= i\epsilon c^b \partial_b c^a,\\ \delta_Q b_{ab} &= i\epsilon T_{ab},\end{aligned}$$

where T_{ab} here is the total stress tensor. Note that, on-shell, $c(z)$ is holomorphic and so the equation of motion $\bar{\partial} c = 0$ tells us that c^a is a conformal Killing vector field (albeit with the wrong statistics). Thus, under the BRST transformation, the metric is invariant, consistent with the gauge-fixing requirement[31].

$$\delta_Q h_{ab} = \epsilon\left(\mathcal{P}c\right)_{ab} = 0.$$

where we have used the fact that c^a is a conformal Killing vector.

14.1 BRST Cohomology and the physical spectrum of the string

What makes the BRST symmetry so useful is that all physical states must be in the cohomology of $\mathcal{Q}_B$. We break this argument down into the following steps. Let $|\phi\rangle$ be a physical state, we can show that:

$|\phi\rangle \in$ **Ker**$(\mathcal{Q}_B)$**:** We require that all physical results derivable from the theory be independent of the choice of gauge. Consider the observable

$$\langle\phi_i|\phi_f\rangle = \int \mathcal{D}\phi\ \phi_i\phi_f\ e^{iS[\phi]+i\{\mathcal{Q}_B,\Psi[b,h]\}}$$

and now consider a change in the gauge-fixing functional, corresponding to a change in the gauge-fixing choice, the change in the correlation function is

$$\begin{aligned}\delta\langle\phi_i|\phi_f\rangle &= \int \mathcal{D}\phi\ \phi_i\phi_f\ e^{iS[\phi]+i\{\mathcal{Q}_B,\Psi[b,h]+\delta\Psi\}} - \int \mathcal{D}\phi\ \phi_i\phi_f\ e^{iS[\phi]+\{\mathcal{Q},\Psi[b,h]\}}\\ &= i\int \mathcal{D}\phi\ \phi_i\phi_f\ \{\mathcal{Q}_B,\delta\Psi\}\ e^{iS[\phi]+i\{\mathcal{Q}_B,\Psi[b,h]\}}\\ &= i\delta\langle\phi_i|\{\mathcal{Q}_B,\delta\Psi\}|\phi_f\rangle\end{aligned}$$

For this to be true for all $\delta\Psi$, we require that all physical states satisfy

$$\mathcal{Q}_B|\phi_f\rangle = 0, \qquad \langle\phi_i|\mathcal{Q}_B = 0$$

[31] This is true of $\mathrm{Diff}_0\times$Weyl transformations. One could allow for moduli transformations to be included also, in which case there will be a residual transformation of the metric under BRST which will involve the moduli only.

If we assume that $\mathcal{Q}_B^\dagger = \mathcal{Q}_B$, then it must be that *all physical states are BRST closed*, i.e.

$$\mathcal{Q}_B|\phi\rangle = 0.$$

$\mathcal{Q}_B^2 = 0$**:** The BRST charge generates a symmetry of the theory and so, by Noether's theorem it must be conserved. This is usually measured by the charge being required to commute with the Hamiltonian of the theory.

$$[\mathcal{Q}_B, H] = 0.$$

We now change the explicit form of the Hamiltonian by changing the gauge choice. Since we demand that such a change does not affect the physics we require that the BRST charge is still conserved under this new Hamiltonian. Since the change is given by

$$\{\mathcal{Q}_B, \delta\Psi\},$$

we require that

$$\begin{aligned} 0 &= [\mathcal{Q}_B, \{\mathcal{Q}_B, \delta\Psi\}] \\ &= -[\delta\Psi, \{\mathcal{Q}_B, \mathcal{Q}_B\}] - [\mathcal{Q}_B, \{\delta\Psi, \mathcal{Q}_B\}] \\ &= -[\delta\Psi, \{\mathcal{Q}_B, \mathcal{Q}_B\}] - [\mathcal{Q}_B, \{\mathcal{Q}_B, \delta\Psi\}] \\ &= -[\delta\Psi, \{\mathcal{Q}_B, \mathcal{Q}_B\}] \end{aligned}$$

using the Jacobi identity in the second line. We see then that, for general $\delta\Psi$, we have

$$\mathcal{Q}_B^2 = \frac{1}{2}\{\mathcal{Q}_B, \mathcal{Q}_B\} = 0.$$

$|\phi\rangle \notin$ **Im(**$\mathcal{Q}_B$**):** Given that $\mathcal{Q}_B^2 = 0$, it is clear that any state of the form $|\xi\rangle = \mathcal{Q}_B|\Lambda\rangle$ is in the kernel of $\mathcal{Q}_B$; however, such states can also be seen to have zero norm

$$\langle\xi|\xi\rangle = \langle\Lambda|\mathcal{Q}_B^2|\Lambda\rangle = 0.$$

Such states are also orthogonal to all physical states

$$\langle\xi|\phi\rangle = \langle\Lambda|\mathcal{Q}_B|\phi\rangle = 0.$$

Such states decouple entirely from the theory, so we take any physical state to be defined up to an arbitrary BRST exact term

$$|\phi\rangle \sim |\phi\rangle + \mathcal{Q}_B|\Lambda\rangle.$$

Given that $|\phi\rangle \in \mathrm{Ker}(\mathcal{Q}_B)$ but $|\phi\rangle \notin \mathrm{Im}(\mathcal{Q}_B)$ we have then, by definition, that the physical states in the theory are in the cohomology of Q

$$|\phi\rangle \in \mathscr{H}_{BRST} = \frac{\mathscr{H}_{\text{closed}}}{\mathscr{H}_{\text{exact}}} = \frac{\mathrm{Ker}(\mathcal{Q}_B)}{\mathrm{Im}(\mathcal{Q}_B)}$$

for $|\phi\rangle$ to be a physical state.

The state operator correspondence tells us that an operator creating a physical state must be BRST invariant i.e. if

$$\lim_{z\to 0}\phi(z)|0\rangle = |\phi\rangle,$$

then, given that the vacuum is BRST invariant,

$$\mathcal{Q}_B|\phi\rangle = 0, \qquad \Longleftrightarrow \qquad [\mathcal{Q}_B, \phi(z)] = 0.$$

14.2 The BRST Charge

We now look for a charge $\mathcal{Q}_B$ that generates the above BRST transformations and is nilpotent, i.e.

$$\mathcal{Q}_B^2 = 0.$$

It will be useful to treat the holomorphic and anti-holomorphic sectors seperately and write

$$\mathcal{Q}_B = Q_B + \bar{Q}_B,$$

and require that

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = 0, \qquad \bar{Q}_B^2 = \frac{1}{2}\{\bar{Q}_B, \bar{Q}_B\} = 0, \qquad \{Q_B, \bar{Q}_B\} = 0$$

On the physical fields $X^\mu(z)$, the BRST transformation is simply the conformal transformation with the parameter $v(z)$ replaced by $\epsilon c(z)$ and so the obvious candidate is

$$Q_B = \oint \mathrm{d}z\; c(z)T_X(z).$$

But this does not reproduce the correct ghost transformations, nor does it satisfy $Q^2 = 0$. It is not hard to show that the BRST charge

$$\boxed{Q_B = \oint \mathrm{d}z\; c(z)\Big(T_X(z) + \frac{1}{2}T_{\mathrm{gh}}(z)\Big)}$$

reproduces the correct BRST transformations.

$$\begin{aligned}
[Q_B, X^\mu(\omega)] &= c(\omega)\partial X^\mu(\omega)\\
\{Q_B, c(\omega)\} &= c(\omega)\partial c(\omega)\\
\{Q_B, b(\omega)\} &= \mathcal{T}(\omega)\\
[Q_B, \mathcal{T}(\omega)] &= \frac{D-26}{12}\partial^3 c(\omega)
\end{aligned}$$

It is easy to check these. The X^μ transformation follows directly from the action of the stress tensors on X^μ. The b-ghost transformation may be calculated as follows

$$\begin{aligned}
\{Q_B, b(\omega)\} &= \oint_{z=0}\frac{\mathrm{d}z}{2\pi i}\left\{c(z)\Big(T_X(z) + \frac{1}{2}T_{\mathrm{gh}}(z)\Big), b(\omega)\right\}\\
&= \oint_{z=\omega}\frac{\mathrm{d}z}{2\pi i}\left(\Big(T_X(z) + \frac{1}{2}T_{\mathrm{gh}}(z)\Big)\frac{1}{z-\omega} + \frac{1}{2}c(z)\left(\frac{2}{(z-\omega)^2}b(\omega) + \frac{1}{z-\omega}\partial b(\omega)\right)\right)
\end{aligned}$$

Writing

$$c(z) = c(\omega) + (z-\omega)\partial c(\omega) + ...$$

$$\begin{aligned}\{Q_B, b(\omega)\} = \oint_{z=\omega} \frac{\mathrm{d}z}{2\pi i}\Bigg(&\frac{1}{z-\omega}\Big(T_X(z) + \frac{1}{2}T_{\text{gh}}(z)\Big) \\ &+\frac{1}{2}\frac{2}{(z-\omega)^2}c(\omega)\partial b(\omega) + \frac{1}{2}\frac{1}{z-\omega}\left(2\partial c(\omega) b(\omega) + c(\omega)\partial b(\omega)\right)\Bigg)\end{aligned}$$

The fist term on the second line doesn't contribute anything and the remaining terms may be written

$$\begin{aligned}\{Q_B, b(\omega)\} &= \oint_{z=\omega} \frac{\mathrm{d}z}{2\pi i}\frac{1}{z-\omega}\left(T_X(z) + \frac{1}{2}T_{\text{gh}}(z) - \partial(b(\omega)c(\omega)) + \frac{1}{2}\partial b(\omega)\, c(\omega)\right) \\ &= \oint_{z=\omega}\frac{\mathrm{d}z}{2\pi i}\frac{\mathcal{T}(\omega)}{z-\omega}\end{aligned}$$

and so

$$\delta_Q b(\omega) = \mathcal{T}(\omega),$$

as required.

14.3 BRST Current and the Conformal Anomaly

It is useful to define the BRST current $j_B(z)$ as

$$\mathcal{Q}_B = Q_B + \bar{Q}_B = \frac{1}{2\pi i}\oint \mathrm{d}z\; j_B(z) - \frac{1}{2\pi i}\oint \mathrm{d}\bar{z}\; \bar{j}_B(\bar{z}).$$

The natural guess is $j_B(z) = c(z)\left(T_X(z) + \frac{1}{2}T_{\text{gh}}(z)\right)$; however, adding terms regular at $z=0$ to the current will not change Q so there is some ambiguity here. This can be resolved by asking that $j_B(z)$ transforms as a vector under conformal transformations. Our first guess does not but a minor modification

$$j_B(z) = c(z)\left(T_X(z) + \frac{1}{2}T_{\text{gh}}(z)\right) + \frac{3}{2}\partial^2 c(z)$$

The $j_B(z)j_B(\omega)$ OPE is given by

$$j_B(z)j_B(\omega) = -\frac{D-18}{2(z-\omega)^3}c\partial c(\omega) - \frac{D-18}{4(z-\omega)^2}c\partial^2 c(\omega) - \frac{D-26}{12(z-\omega)}c\partial^3 c(\omega) + ...$$

We can check $Q^2 = 0$ is consistent with the OPE calculations above

$$\begin{aligned}\{Q_B, Q_B\} &= \oint_{z=0}\frac{\mathrm{d}z}{2\pi i}\oint_{\omega=0}\frac{\mathrm{d}\omega}{2\pi i}\,\{j_B(z), j_B(\omega)\} \\ &= \oint_{z=0}\frac{\mathrm{d}z}{2\pi i}\oint_{\omega=z}\frac{\mathrm{d}\omega}{2\pi i}\left(-\frac{D-18}{2(z-\omega)^3}c\partial c(\omega) - \frac{D-18}{4(z-\omega)^2}c\partial^2 c(\omega) - \frac{D-26}{12(z-\omega)}c\partial^3 c(\omega)\right) \\ &= \frac{D-26}{12}\oint_{z=0}\frac{\mathrm{d}z}{2\pi i}c(z)\partial^3 c(z)\end{aligned}$$

as all of the $D-18$ contributions cancel out, which may be seen by integrating by parts. The remaining contour integral is not zero so, in order to have $Q_B^2 = 0$, we require $D = 26$, as above. A similar result holds for $\bar{Q}_B$

Part IV

Symmetry Enhancement and T-Duality

15 Strings on Tori

16 Symmetry Enhancement

Something very special happens when the radius of the target space circle is exactly

$$R = \sqrt{\alpha'}.$$

16.1 The $SU(2)$ Operator Algebra

$$J^3(z) = \partial X_L(z), \qquad J^{\pm}(z) =: \exp\left(i2X_L(z)/\sqrt{\alpha'}\right) :$$

Using standard OPE techniques (try it!) we find the operator product expansions are

$$H(z)E^{\pm}(w) = ...$$

From this we can define the charges[32]

$$H = \oint \mathrm{d}z J^3(z), \qquad E^{\pm} = \oint \mathrm{d}z J^{\pm}(z).$$

Remarkably we find that these generate an $SU(2)$ Lie algebra!

16.2 The Target Space Perspective: A Stringy Higgs Mechanism

17 Unbroken Symmetry: T-Duality

[32]the currents we are integrating are wight (1,0) which means the charges will be conserved - can you see why?

Part V

Scattering Amplitudes

18 What's The Big Idea?

What should we calculate with this theory we have built? We only have a perturbative understanding of the theory and we would like to understand its connection with conventional field theory. As such it makes sense to see what the S-Matrix of string theory looks like and learn about the perturbative similarities and differences with conventional field theory. After all, one of the motivations for considering the theory in the first place were the divergences present in conventional field theory approaches to gravity. It is important to see how string theory overcomes these problems.

Another motivation is that it is very difficult to think up local diffeomorphism invariant quantities that could act as observables. The S-Matrix includes quantities living 'at infinity', where we take local diffeomorphism transformations to die off and so it makes sense to look at correlation functions of such objects.

One of the more remarkable features of string theory is that one can calculate the amplitudes for scattering processes in spacetime by computing correlation functions on the worldsheet. The aim of this chapter is to explain how this is done. Recall the infinite cylinder describing a free string propagating from the infinite past to the infinite future. The conformal map

$$z = e^{\tau + i\sigma},$$

is defined globally and maps the cylinder to the complex plane. The point at the origin is the location of the string in the infinite past. For an initial string at time $\tau = \tau_i$, the initial string is the boundary of a small disc $\mathscr{D}_i$, centred on the origin that is removed. As $\tau_i \to -\infty$, $\mathscr{D}_i$ shrinks to a point and the worldsheet is the complex plane with the origin removed. The same argument holds for the final state string at $\tau_f \to \infty$, where we remove the point at infinity. Alternatively, and the mental picture we shall often adopt, is to thinlkof this as the Riemann sphere with two points removed. We often call these removed points *punctures*. The initial and final states are then encoded as operators inserted at the punctures. The idea is that the operator $V_i(z, \bar{z})$ that creates the initial state $|i\rangle$ is given by the state/operator correspondence

$$|i\rangle = \lim_{z,\bar{z}\to 0} V_i(z, \bar{z})|0\rangle.$$

For example, for the graviton we shall find

$$\varepsilon_{\mu\nu}\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle = \lim_{z,\bar{z}\to 0} \varepsilon_{\mu\nu} : \partial X^{\mu}(z)\bar{\partial}X^{\nu}(\bar{z})e^{ik\cdot X(z,\bar{z})} : |0\rangle.$$

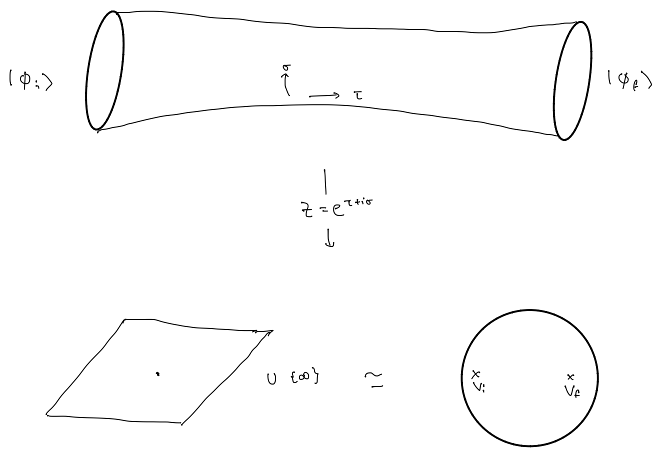


Figure 12. The Cylinder may be mapped from to the complex plane with the origin removed. Alternatively, we may view it as the Riemann sphere with two punctures.

When we first discussed the path integral we introduced the expression

$$\langle \Psi_i | \Psi_f \rangle = \int_i^f \mathcal{D}X\mathcal{D}h \, e^{-S[X,h]},$$

without really thinking abut how we specify the initial and final states. We se that the vertex operators give a natural way to specify asymptotic states in terms of operator insertions so that

$$\langle \Psi_i | \Psi_f \rangle = \int \mathcal{D}X\mathcal{D}h \, V_i V_f \, e^{-S[X,h]}.$$

This is all fine for the cylinder, where the conformal map described above is globally defined but what of more complicated Riemann surfaces with many initial and final states? The idea is a local take on the global construction described above. Take the worldsheet and remove the cylinders that contain the initial and final states. On this cylinder we define local a local coordinate system and a map from the cylinder to a local coordinate system on an annulus. Taking the initial state to the infinite past gives a map from the semi-infinite cylinder to a punctured disc where, by convention, the puncture is at the origin of the local coordinate system. We then glue these punctured discs back into the worldsheet to get an Riemann surface with many punctures, each with a local coordinate system centred on the puncture. The details of this construction are not needed for what we consider here but the construction generalises straightforwardly to higher genus Riemann surfaces.

Higher genus Riemann surfaces correspond to loop corrections in spacetime. For example, we associate the genus one worldsheets with one-loop contributions to spacetime processes.

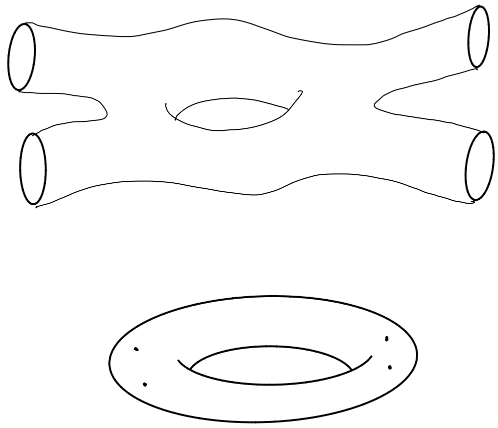


Figure 13. The one-loop correction to the propagator may be mapped to a torus with two punctures.

It is a remarkable feature of string theory that spacetime processes may be computed using two-dimensional conformal field theory.

19 Preliminaries

There are a couple of things we need to discuss before computing scattering amplitudes.

19.1 Ghost Vacua

We start with the standard vacuum $|0\rangle$. This vacuum is invariant under the $SL(2;\mathbb{C})$ sub-algebra of the Virasoro algebra generated by $\{L_0, L_\pm\}$ so we shall call it the "$SL(2;\mathbb{C})$-invariant vacuum" to distinguish it from the other vacua we shall need to introduce. The state-operator correspondence gives

$$|\Phi\rangle = \lim_{z\to 0} \Phi(z)|0\rangle.$$

What are the conditions needed for this limit to exist? We may write this as

$$|\Phi\rangle = \lim_{z\to 0} \sum_n \phi_n z^{-n-h}|0\rangle.$$

Clearly those terms for which $n < -h$ vanish in the limit. The $n = -h$ term has no z-dependence and so is not affected by the limit, leaving

$$|\Phi\rangle = \lim_{z\to 0} \left(\phi_{-h} + \sum_{n>-h} \phi_n z^{-n-h} \right) |0\rangle.$$

A necessary condition for the limit to exist is that

$$\boxed{\phi_n|0\rangle = 0, \qquad n > -h.}$$

The state is then $|\Phi\rangle = \phi_{-h}|0\rangle$. This nicely reproduces the requirement that

$$\lim_{z\to 0} \partial X^\mu(z)|0\rangle = \alpha^\mu_{-1}|0\rangle$$

and $\alpha^n|0\rangle = 0$ for $n > -1$. For the ghost field we then have the conditions that

$$b_n|0\rangle = 0, \qquad n > -2 \qquad \text{and} \qquad c_n|0\rangle = 0, \qquad n > +1.$$

where

$$b(z) = \sum_n b_n z^{-n-2}, \qquad c(z) = \sum_n c_n z^{-n+1}.$$

These are not the usual conditions we expect from a vacuum state. One usually would expect all non-negative modes to annihilate the vacuum. What is going on?

In fact $|0\rangle$, though the vacuum for the $X^\mu(z)$ sector is not the vacuum for the ghost sector. Indeed, a curious feature of the ghost system is that there is not a unique vacuum. If we define the vacuum as the state which is annihilated by the operators c_n and b_n for $n \geq 0$ we find, the failure of all such operators to commute means that there is no unique vacuum that can satisfy both of these conditions. Instead we are left with two distinct vacua, which we shall call $|\uparrow\rangle$ and $|\downarrow\rangle$. We define $|\uparrow\rangle$ by

$$c_0|\uparrow\rangle = 0.$$

Since b_0 and c_0 don't commute, they cannot both annihilate $|\uparrow\rangle$ (they would then both have it is a zero-eigenvalue eigenstate). It must be that b_0 acting on $|\uparrow\rangle$ gives another state. Let us call this state $|\downarrow\rangle$

$$b_0|\uparrow\rangle = |\downarrow\rangle.$$

It then follows that

$$b_0|\downarrow\rangle = b_0^2|\uparrow\rangle = 0$$

and

$$c_0|\downarrow\rangle = c_0 b_0|\uparrow\rangle = (\{c_0, b_0\} - b_0 c_0)|\uparrow\rangle = |\uparrow\rangle$$

thus we have

$$\boxed{c_0|\uparrow\rangle = 0, \qquad b_0|\downarrow\rangle = 0, \qquad b_0|\uparrow\rangle = |\downarrow\rangle, \qquad c_0|\downarrow\rangle = |\uparrow\rangle, \qquad b_0|\uparrow\rangle = |\uparrow\rangle}$$

We can then deduce that[33] they are related by

$$c_0|\downarrow\rangle = |\uparrow\rangle, \qquad b_0|\uparrow\rangle = |\downarrow\rangle.$$

[33]

$$c_0|\downarrow\rangle = c_0 b_0|\uparrow\rangle = (1 - b_0 c_0)|\uparrow\rangle = |\uparrow\rangle.$$

and similarly for the second relationship.

An immediate question arises; what is the relationship between these ghost vacua and the $SL(2;\mathbb{C})$-inarviant vacuum $|0\rangle$? If we want to think of $|\uparrow\rangle$ as a c-ghost vacuum, then we might also like $c_1|\uparrow\rangle = 0$. This condition is satisfied if we choose

$$c_1|0\rangle = |\downarrow\rangle$$

so that

$$\boxed{|\uparrow\rangle = c_0 c_1|0\rangle, \qquad |\downarrow\rangle = c_1|0\rangle}$$

Other considerations, such as ghost number, also lead us to this conclusion.

We can consider inner products of these vacua:

$$\langle\uparrow|\uparrow\rangle = \langle 0|c_{-1}c_0^2 c_1|0\rangle = 0, \qquad \langle\downarrow|\downarrow\rangle = \langle\uparrow|b+0^2|\uparrow\rangle = 0,$$

However $\langle\uparrow|\downarrow\rangle$ is not vanishing

$$\langle\uparrow|\downarrow\rangle = \langle\uparrow|\downarrow\rangle = \langle 0|c_{-1}c_0c_1|0\rangle \neq 0.$$

We have a enough flexibility left in the definitions of these vacua to chose the normalisation

$$\boxed{\langle 0|c_{-1}c_0c_1|0\rangle = 1} \tag{19.1}$$

The Hilbert space factorises into embedding and ghost sectors $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_{b,c}$. Reflecting this, the vacuum may be written as

$$|0,\downarrow\rangle = |0\rangle_X|\downarrow\rangle$$

where we have chosen $|\downarrow\rangle$ to be the ghost vaccum.

Note that

$$\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = \langle 0|c_{-1}c_0c_1|0\rangle \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix} = (z_1-z_2)(z_2-z_3)(z_3-z_1).$$

and so

$$\boxed{\left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle = |z_1-z_2|^2|z_2-z_3|^2|z_3-z_1|^2.}$$

This result will be useful in the computation of scattering amplitudes.

19.2 The Dilaton and the String Coupling

We can imagine the string propagating in a background with metric $g_{\mu\nu}(X)$, B-field $B_{\mu\nu}(X)$ and dilaton $\Phi(X)$. The metric and B-field couple to the string as

$$S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}h^{ab}g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu - \frac{i}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}\epsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu$$

and there are conditions on $g_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ such that the theory is conformally invariant.

Suppose we have a background dilaton field $\Phi(X)$. This can be coupled to the string worldsheet by adding to the action the term

$$S_\Phi[X,h] = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}\Phi(X)R_\Sigma$$

A few comments are in order

- This breaks classical Weyl invariance. However, the term enters at higher order in α' that the other terms in the action and so a proper analysis requires considering loop effects on the worldsheet. One finds that the quantum theory is actually still Weyl invariant provided the background metric and dilaton obey the appropriate equations of motion. We will discuss this briefly in the following chapter.

- If the dilaton is evaluated at an expectation value $\langle\Phi(X)\rangle = \Phi_0$, then the this term in the action becomes a topological term

  $$S_\Phi[h] = \Phi_0\chi = \Phi_0(2g-2)$$

  where χ is the Euler character of the Riemann surface and g is its genus.

We define the closed string coupling as

$$g_c = e^{\Phi_0}$$

and the contribution to the path integral is then

$$\mathcal{A}_n = \sum_{g=0}^{\infty} g_c^{2g-2}\frac{1}{|CKG|}\int_{\mathcal{M}_g}\mathrm{d}^s t\int \mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c}\prod_{i=1}^{s}(\mu_i|b)(\bar{\mu}_i|\bar{b})\, e^{-S_{\mathrm{gh}}[b,c]}\,\langle V_1V_2...V_n\rangle$$

where we have interpreted the path integral over the worldsheet metric to include a sum over all genus.

Imagine a closed string being emitted and reabsorbed by a worldsheet of genus g. This adds a handle to the surface and so raises the genus by one $g\to g+1$ and so adds in an extra g_c^2 to the amplitude. As such we associate a factor of g_c with the emission or absorption of a closed string. It is in this sense that we think of g_c as a *coupling constant* for the closed string theory.

20 Vertex Operators

We are looking for operators that are in the BRST cohomology. We shall see that there are two inequivalent ways to represent the same state. Suppose we construct an operator $\phi(z,\bar{z})$ that satisfies

$$[Q,\phi] = \partial(c\phi), \qquad [\bar{Q},\phi] = \bar{\partial}(\bar{c}\phi)$$

where $\mathcal{Q}_B = Q_B + \bar{Q}_B$ is the BRST charge. Then, we can construct BRST-invariant integrals

$$V_\phi = \int_\Sigma \mathrm{d}^2 z\ \phi(z, \bar{z}),$$

since the BRST operator gives a surface term when commuted with the integrand which vanishes in the integral. A problem with such an object is that it is manifestly non-local; however, we shall be able to make sense of this.

An alternative operator is given by

$$U(z, \bar{z}) = c(z)\bar{c}(\bar{z})\phi(z, \bar{z})$$

This is clearly BRST closed

$$[Q, c\bar{c}\phi] = (c\partial c)\ \bar{c}\ \phi + c\bar{c}\ \partial(c\phi) = 0.$$

and so, given an appropriate $\phi(z, \bar{z})$, we have two types of operator in the cohomology.

How easy is it to find $\phi(z, \bar{z})$ that transforms in the correct way? We expect the field to be a primary operator, so that under conformal transformations we have

$$\delta_v \phi(z, \bar{z}) = h_\phi \partial v(z) + v(z)\partial\phi(z, \bar{z}) + \text{c.c.}.$$

thus, under BRST transformations, we require

$$\begin{aligned}[][Q_B, \phi(z, \bar{z})] &= h_\phi \partial c(z)\, \phi(z, \bar{z}) + c(z)\partial\phi(z, \bar{z}) \\ &= (h_\phi - 1)\partial c(z)\, \phi(z, \bar{z}) + \partial(c(z)\phi(z, \bar{z}))\end{aligned}$$

where we have integrated by parts to get to the second line. It is clear that if

$$(h_\phi, \bar{h}_\phi) = (1, 1),$$

then $\phi(z, \bar{z})$ will transform in the required way.

An observation: Before we gauge fix the metric we do not have ghosts and so we only have the non-local integrated vertex operators, consistent with a quantum theory of gravity. Once we gauge fix gravity and no longer work in a diffeomorphism invariant framework, we have ghosts and we can consistently construct local vertex operators using these ghosts.

We now consider some examples:

20.1 The Tachyon

The simplest boundary condition we can put on a string is that the worldsheet must pass through a particular point, i.e. the point $(z_i, \bar{z}_i)$ on the worldsheet should correspond to the point x_i^μ in spacetime. We might write this as a delta functional

$$\delta[x_i^\mu - X^\mu(z_i, \bar{z}_i)]$$

to be inserted into the path integral. We are more interested in computing scattering amplitudes in momentum space, so we Fourier transform this to momentum space

$$\int \mathrm{d}^{26}k_i\, e^{ik_i\cdot x_i}\,\delta[x_i^\mu - X^\mu(z_i,\bar{z}_i)] = e^{ik\cdot X^\mu(z,\bar{z})}.$$

We know that the conformal weight of this operator is

$$(h,\bar{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right).$$

so, if

$$\frac{\alpha' k^2}{4} = 1,$$

i.e. we are dealing with the Tachyon, with mass $m^2 = -k^2 = -4/\alpha'$ then we can construct an integrated and an unintegrated vertex operator

$$\boxed{U_T(z,\bar{z}) = \frac{2g_c}{\alpha'} : c(z)\bar{c}(\bar{z})e^{ik\cdot X(z,\bar{z})} :, \qquad V_T = \frac{2g_c}{\alpha'}\int_\Sigma \mathrm{d}^2 z\ : e^{ik\cdot X(z,\bar{z})} : .}$$

where conventional factors of the coupling constant have been added in.

20.2 Massless Modes

We can get a feel for what might constitute the graviton vertex operator by considering a linear perturbation of the metric in the path integral. Let $\eta_{\mu\nu} \to g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}e^{ik\cdot X(z,\bar{z})}$ by a plane wave deformation of the metric. The X part of the path integral becomes

$$\int \mathcal{D}X\ \exp\left(-T\int_\Sigma \mathrm{d}^2z\ g_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu\right)$$
$$= \int \mathcal{D}X\ e^{-S_\eta[X]}\ \left(1 - T\int_\Sigma \mathrm{d}^2z\ h_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ik\cdot X(z,\bar{z})} + \ldots\right)$$

where

$$S_\eta[X] = -T\int_\Sigma \mathrm{d}^2z\ g_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu$$

It is clear that the leading order correction to the flat space metric is given by inserting the operator

$$T\int_\Sigma \mathrm{d}^2z\ h_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ik\cdot X(z,\bar{z})}$$

into the path integral. This suggests the following massless vertex operators

Dilaton

$$U_\phi(z,\bar{z}) = \frac{2g_c}{\alpha'} : c(z)\bar{c}(\bar{z})\partial X^\mu\bar{\partial}X_\mu e^{ik\cdot X(z,\bar{z})} :, \qquad V_\phi = \frac{2g_c}{\alpha'}\int_\Sigma \mathrm{d}^2z\ \phi\eta_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ik\cdot X(z,\bar{z})}$$

Graviton

$$U_g(z,\bar{z}) = \frac{2g_c}{\alpha'} h_{\mu\nu} : c(z)\bar{c}(\bar{z})\partial X^{(\mu}\bar{\partial}X^{\nu)} e^{ik\cdot X(z,\bar{z})} : \qquad V_g = \frac{2g_c}{\alpha'}\int_\Sigma \mathrm{d}^2 z\; h_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X(z,\bar{z})}$$

B**-Field**

$$U_B(z,\bar{z}) = \frac{2g_c}{\alpha'} b_{\mu\nu} : c(z)\bar{c}(\bar{z})\partial X^{[\mu}\bar{\partial}X^{\nu]} e^{ik\cdot X(z,\bar{z})} : \qquad V_B = \frac{2g_c}{\alpha'}\int_\Sigma \mathrm{d}^2 z\; b_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X(z,\bar{z})}$$

These all have the correct weight if $k^2 = 0$.

20.3 A comment on massive modes

We could clearly continue and explore the vertex operators for massive modes. One reason not to do this is that the vertex operators we already have will be more than enough to keep us busy in this course. Another reason not to is that massive modes are subject to mass renormalisation. This is a finite shift in the mass so no infinities develop but it does mean that the physical vertex operators have a mass which means we cannot deal with them easily in the on-shell first quantised formalism. This issue can be overcome. One way to do so is to use a second quantised string field theory. This would take us far beyond what we want to cover in this course and as stated above, we have more than enough to be getting on with. Thus, and with some regret, we shall not consider the massive modes further in this course.

21 The general structure of the S-Matrix

In terms of the ghost fields, this S-matrix element $\langle V(z_1)V(z_2)...V(z_n)\rangle$ may be written as

$$\mathcal{A}_n = \sum_{g=0}^{\infty} g_c^{2g-2}\frac{1}{|CKG|}\int_{\mathcal{M}} \mathrm{d}^s m \int \mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c}\prod_{i=1}^{s}(\mu_i|b)(\bar{\mu}_i|\bar{b})\, e^{-S_{\mathrm{gh}}[b,c]}\,\langle V_1V_2...V_n\rangle \prod_{i,a} c^a(\hat{\sigma}_i)$$

where we define

$$\langle V_1V_2...V_n\rangle_X = \int \mathcal{D}X\, V_1V_2...V_n\, e^{-S[X]}$$

and the $V(z)$ are vertex operators which may be written as

$$V_i = g_c\int \mathrm{d}^2 z_i\, \phi(z_i,\bar{z}_i)$$

21.1 Tree Level

At tree level $g = 0$ and the moduli space is a point, so $s = 0$ and the conformal Killing group is $SL(2;\mathbb{C})$, so we have

$$\mathcal{A}_n = \frac{1}{|SL(2,\mathbb{C})|} g_c^{n-2}\int\prod_{i=1}^{n}\mathrm{d}^2 z_i\left\langle\prod_{i=1}^{3}c(z_i)\bar{c}(\bar{z}_i)\right\rangle_{b,c}\left\langle\phi(z_1)\phi(z_2)...\phi(z_n)\right\rangle_X$$

where the ghost and matter contribution factorise

$$\Big\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \Big\rangle_{b,c} = \int \mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c}\, e^{-S_{\rm gh}[b,c]} \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i),$$

$$\Big\langle \phi(z_1)\phi(z_2)...\phi(z_n) \Big\rangle_X = \int \mathcal{D}X\; e^{-S[X]}\; \phi(z_1)\phi(z_2)...\phi(z_n).$$

21.1.1 $SL(2;\mathbb{C})$

The infinitesimal action of $SL(2;\mathbb{C})$ on the coordinate z_i may be written as

$$z_i \to z_i' = a_1 + a_2 z_i + a_3 z_i^2,$$

where the a_i are infinitesimal parameters. We can relate integration over z_i to integration over a_i as

$$\det(\partial z_i/\partial a_j)\, \mathrm{d}^3 a = \mathrm{d}^3 z$$

where the Jacobian is

$$\det(\partial z_i/\partial a_j) = \begin{vmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_2 & z_3^2 \end{vmatrix} = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$

and so the $SL(2;\mathbb{C})$ volume element is

$$\mathrm{d}^2 a_1\, \mathrm{d}^2 a_2\, \mathrm{d}^2 a_3 = \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3}{|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^2}$$

which may be written as

$$\frac{1}{d|SL(2;\mathbb{C})|} = \frac{|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^2}{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3} = \frac{\langle\, \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i)\,\rangle}{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3}$$

We can now see that the $\langle c(z_1)c(z_2)c(z_3)\rangle$ factor we included dealt with the Jacobian, all that remains is to factor out by $\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3$.

Thus we interpret the quotient by $SL(2;\mathbb{C})$ to have the effect of removing three of the integrals (i.e. fixing the locations of three punctures). Any three will do, but for definiteness we choose the first three

$$\mathcal{A}_n = g_c^{n-2} \Big\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \Big\rangle_{b,c} \int \prod_{i=4}^{n} \mathrm{d}^2 z_i\, \Big\langle \phi(z_1)\phi(z_2)...\phi(z_n) \Big\rangle_X$$

where we are at liberty to fix z_1, z_2 and z_3 to any distinct points. The usual choice is 0, 1 and ∞, but any will do and the $SL(2;\mathbb{C})$ invariance of the metric means that the amplitude does not depend on the choice we make.

21.1.2 The Scattering Amplitude Simplified

Notice that the expression for the amplitude could be written as

$$\boxed{\mathcal{A}_n = g_c^{n-2}\Big\langle \prod_{i=1}^{3} U_i(z_i,\bar{z}_i)\; V_4...V_n \Big\rangle}$$

where the U_i are un-intergrated vertex operators and the V_i are integrated vertex operators.

22 Some Tree-Level Amplitudes using Path Integrals

From now on we work in the complex plane. We want to know how to compute the correlation function

$$\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle_X = \int \mathcal{D}X\; e^{-S[X]}\phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n).$$

We start by introducing the source term

$$S_J[X,J] = \int_\Sigma \mathrm{d}^2 z\; J_\mu X^\mu.$$

and considering the normalised generating functional

$$Z[J] = Z^{-1}[0]\int \mathcal{D}X\; e^{-S[X]-S_J[X,J]}.$$

It is useful to separate the constant centre of mass term from the X^μ, so we write

$$X^\mu(z) = x^\mu + \tilde{X}^\mu(z), \qquad \int \mathcal{D}X = \int \mathrm{d}^{26}x \int \mathcal{D}\tilde{X}.$$

Introducing the Greens function

$$-\frac{1}{\pi\alpha'}\Box G(z,\omega) = \delta^2(z-\omega),$$

i.e.

$$G(z,\omega) = -\frac{\alpha'}{2}\ln|z-\omega|^2$$

we may write the modified action as

$$\begin{aligned} S[X]+S_J[X,J] &= \frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2 z\; \tilde{X}^\mu \Box \tilde{X}^\nu \eta_{\mu\nu} + \int_\Sigma \mathrm{d}^2 z\; J_\mu \tilde{X}^\mu + x^\mu \int_\Sigma \mathrm{d}^2 z\; J_\mu(z) \\ &= \frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2 z\; Y^\mu(z)\Box Y_\mu(z) + \frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\omega\; J^\mu(z) G(z,\omega) J_\mu(\omega) \\ &\quad + x^\mu \int_\Sigma \mathrm{d}^2 z\; J_\mu(z) \end{aligned}$$

where

$$Y^\mu(z) = \tilde{X}^\mu(z) - \int_\Sigma \mathrm{d}^2\omega\; G(z,\omega)J^\mu(\omega)$$

and we have completed the square in X^μ in going from the first to the second line. The relationship between X^μ and Y^μ is linear and so we may set $\mathcal{D}X = \mathcal{D}Y$ and first term, quadratic in Y^μ, may be integrated out. We note that the Y^μ dependent terms may be written, up to a factor coming from the zero modes, as

$$Z[0] = \int \mathcal{D}Y\; \exp\left(\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2z\; Y^\mu(z)\Box Y_\mu(z)\right),$$

and so the generating functional becomes

$$Z[J] = \exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2z\,\mathrm{d}^2\omega\; J^\mu(z)G(z,\omega)J_\mu(\omega)\right)\int \mathrm{d}^{26}x\; \exp\left(x^\mu\int_\Sigma \mathrm{d}^2z\; J_\mu(z)\right)$$

Correlation functions may then be computed by functional differentiation

$$\langle X^{\mu_1}(z_1)X^{\mu_2}(z_2)...X^{\mu_n}(z_n)\rangle = \frac{\delta}{\delta J_{\mu_1}(z_1)}\frac{\delta}{\delta J_{\mu_2}(z_2)}...\frac{\delta}{\delta J_{\mu_n}(z_n)}\,Z[J]$$

We shall have no need of this here.

23 Some sample calculations using path integrals

We look at some concrete examples.

23.1 Tachyon Scattering

The X-correlation function is of the form

$$\begin{aligned}\left\langle e^{ik_1\cdot X(z_1,\bar{z}_1)}...e^{ik_1\cdot X(z_1,\bar{z}_1)}\right\rangle_X &= \int \mathcal{D}X\; e^{-S[X]}\; e^{ik_1\cdot X(z_1,\bar{z}_1)}...e^{ik_1\cdot X(z_1,\bar{z}_1)} \\ &= \int \mathcal{D}X\; \exp\left(-S[X] + i\sum_{j=1}^n k_j\cdot X(z_j,\bar{z}_j)\right)\end{aligned}$$

since in the path integral the $e^{ik\cdot X(z,\bar{z})}$ with *functions* not operators. We may write this as

$$\left\langle e^{ik_1\cdot X(z_1,\bar{z}_1)}...e^{ik_1\cdot X(z_1,\bar{z}_1)}\right\rangle_X = \int \mathcal{D}X\; \exp\left(-S[X] - \int_\Sigma \mathrm{d}^2z\; J\cdot X\right)$$

where

$$J(z,\bar{z}) = -i\sum_{i=1}^n k_i\delta^2(z-z_i)$$

Then

$$\int_\Sigma \mathrm{d}^2z\; J_\mu(z) = -i\int_\Sigma \mathrm{d}^2z\;\sum_{i=1}^n k_i\delta^2(z-z_i) = -i\sum_{i=1}^n k_i$$

and so

$$\int \mathrm{d}^{26}x \ \exp\left(x^\mu \int_\Sigma \mathrm{d}^2 z \ J_\mu(z)\right) = \int \mathrm{d}^{26}x \ \exp\left(-ix_\mu \sum_{i=1}^n k_i^\mu\right) = (2\pi)^{26}\delta^{26}\left(\sum_{i=1}^n k_i^\mu\right)$$

and so this delta function enforces overall momentum conservation.

We also have

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\omega \ J^\mu(z) G(z,\omega) J_\mu(\omega) \\
&= -\frac{1}{2}\sum_{i\neq j}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\omega \ k_i^\mu \delta^2(z-z_i) \ G(z,\omega) \ k_{j\mu}\delta^2(\omega - z_j) \\
&= -\frac{1}{2}\sum_{i\neq j} k_i^\mu \Big(-\frac{\alpha'}{2}\ln|z_i - z_j|^2\Big) k_{j\mu}
\end{aligned}$$

so that

$$\begin{aligned}
\exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\omega \ J^\mu(z) G(z,\omega) J_\mu(\omega)\right) &= \prod_{i\neq j} |z_i - z_j|^{\alpha' k_i\cdot k_j/2} \\
&= \prod_{i>j} |z_i - z_j|^{\alpha' k_i \cdot k_j}
\end{aligned}$$

We then have

$$\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle_X = (2\pi)^{26}\delta^{26}\left(\sum_{i=1}^n k_i^\mu\right) \times \prod_{i<j} |z_i - z_j|^{\alpha' k_i\cdot k_j}$$

23.1.1 Three-Point Tachyon Amplitude

We have

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2}\Big((k_1+k_2)^2 - k_1^2 - k_2^2\Big) = \frac{\alpha'}{2}(k_3^2 - k_1^2 - k_2^2)$$

using the mass shell condition for Tachyons $k^2 = -m^2 = 4/\alpha'$, we find

$$\alpha' k_1 \cdot k_2 = -2$$

and so

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle_X = (2\pi)^{26}\delta^{26}\left(\sum_{i=1}^3 k_i^\mu\right) \times \frac{1}{|(z_1-z_2)(z_2-z_3)(z_3-z_1)|^2}$$

which cancels with the ghost contirbution

$$\Big\langle \prod_{i=1}^3 c(z_i)\bar{c}(\bar{z}_i)\Big\rangle_{b,c} = |(z_1-z_2)(z_2-z_3)(z_3-z_1)|^2$$

to give

$$\mathcal{A}_3 = g_c (2\pi)^{26}\delta^{26}\left(\sum_{i=1}^3 k_i^\mu\right)$$

23.1.2 Four-Point Tachyon Amplitude: The Virasoro-Shapiro Amplitude

For the four point amplitude we use the $SL(2;\mathbb{C})$ invariance to choose

$$z_1 = 0, \qquad z_2 = 1, \qquad z_3 = \lambda \to \infty$$

and set $z_4 = z$. The amplitude includes

$$\prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j} = |z_1 - z_2|^{\alpha' k_1 \cdot k_2} |z_1 - z_3|^{\alpha' k_1 \cdot k_3} |z_1 - z_4|^{\alpha' k_1 \cdot k_4} |z_2 - z_3|^{\alpha' k_2 \cdot k_3} |z_2 - z_4|^{\alpha' k_2 \cdot k_4} |z_3 - z_4|^{\alpha' k_3 \cdot k_4}$$

The ghost contributions give $|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2$, giving

$$\begin{aligned}
&\left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle_{b,c} \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j} \\
&= |z_1 - z_2|^{\alpha' k_1 \cdot k_2 + 2} |z_1 - z_3|^{\alpha' k_1 \cdot k_3 + 2} |z_1 - z_4|^{\alpha' k_1 \cdot k_4} |z_2 - z_3|^{\alpha' k_2 \cdot k_3 + 2} |z_2 - z_4|^{\alpha' k_2 \cdot k_4} |z_3 - z_4|^{\alpha' k_3 \cdot k_4} \\
&= |\lambda|^{\alpha' k_1 \cdot k_3 + 2} |z|^{\alpha' k_1 \cdot k_4} |1 - \lambda|^{\alpha' k_2 \cdot k_3 + 2} |1 - z|^{\alpha' k_2 \cdot k_4} |\lambda - z|^{\alpha' k_3 \cdot k_4}
\end{aligned}$$

z takes values in a patch of the Riemann sphere that does not include the point at infinity so we assume that $\lambda \gg 1, z$ and so in the limit $\lambda \to \infty$

$$\left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle_{b,c} \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j} = |\lambda|^{\alpha'(k_1 + k_2 + k_4) \cdot k_3 + 4} |z|^{\alpha' k_1 \cdot k_4} |1 - z|^{\alpha' k_2 \cdot k_4}$$

We note that

$$\alpha'(k_1 + k_2 + k_4) \cdot k_3 + 4 = -\alpha' k_3^2 + 4 = 0.$$

where momentum conservation and the mass-shell condition have been used. The amplitude is then

$$\mathcal{A}_4 = g_c^2 (2\pi)^{26} \delta^{26} \left(\sum_{i=1}^{4} k_i^\mu \right) \int \mathrm{d}^2 z \; |z|^{\alpha' k_1 \cdot k_4} |1 - z|^{\alpha' k_2 \cdot k_4}$$

where we are integrating over the complex plane with the points at $z = 0, 1$ removed (or the Riemann sphere with $z = 0, 1, \infty$ removed).

We can introduce the Madelstam variables

$$t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2.$$

Using the fact that

$$\alpha' k_1 \cdot k_4 = -\frac{\alpha' u}{2} - 4, \qquad \alpha' k_2 \cdot k_4 = -\frac{\alpha' t}{2} - 4$$

, the amplitude may be written as

$$\mathcal{A}_4 = g_c^2 (2\pi)^{26} \delta^{26} \left(\sum_{i=1}^{4} k_i^\mu \right) \int \mathrm{d}^2 z \; |z|^{-\alpha' u/2 - 4} |1 - z|^{-\alpha' t/2 - 4}$$

A few comments are in order:

- We can work harder and write the amplitude in terms of the Gamma functions
$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, \mathrm{d}x.$$
We find the elegant result
$$\mathcal{A}_4 = g_c^2 (2\pi)^{26} \delta^{26} \left(\sum_{i=1}^{4} k_i^\mu \right) 2\pi \frac{\Gamma(\alpha(s))\Gamma(\alpha(t))\Gamma(\alpha(u))}{\Gamma(\alpha(t)+\alpha(u))\Gamma(\alpha(s)+\alpha(u))\Gamma(\alpha(s)+\alpha(t))} \tag{23.1}$$
where
$$\alpha(s) = -1 - \frac{\alpha'}{4} s,$$
and s, t, u are the famed Mandelstam variables
$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2.$$

- If we allow the punctures to coincide then the integrand will have poles at $z = 0, 1, \infty$. The corresponds to the Riemann surface degenerating and physically the mode propagating through the degeneration point is going on-shell.

The expression (23.1) makes the symmetry between the s, t and u channels apparent. Historically, it was of great interest finding scattering amplitudes that has his symmetry (or duality). Indeed it was the investigation of such 'dual models' that lead to some of the first steps in the construction of the bosonic string. Today, we see the duality between these channels as a manifestation of invariance of the path integral under deformations of the worldsheet.

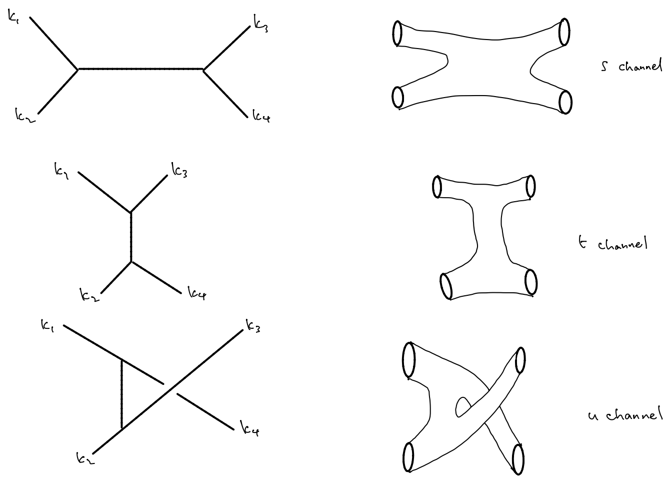


Figure 14. The 'duality' between s, t and u channels.

23.2 Scattering of massless states

We are now interested in vertex operators of the form

$$V_j = g_c \int_\Sigma \mathrm{d}^2z\ \varepsilon_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik_j\cdot X}$$

We can write

$$i\partial X^\mu(z_j)\, e^{ik_j\cdot X(z_j)} = \left[\frac{\partial}{\partial\rho_j}\exp\left(i\sum_{j=1}^n\int_\Sigma \mathrm{d}^2z\left(k_{j\mu}+\rho_j\frac{\partial}{\partial z}\right)X^\mu(z)\delta^2(z-z_j)\right)\right]_{\rho_j=0}$$

similarly, we can write

$$\begin{aligned}&\varepsilon_{\mu\nu}\partial X^\mu(z_j)\bar{\partial}X^\nu(z_j)\, e^{ik_j\cdot X(z_j)}\\ &= -\varepsilon_{\mu\nu}\left[\frac{\partial^2}{\partial\rho_j\partial\bar{\rho}_j}\exp\left(i\int_\Sigma \mathrm{d}^2z\ \delta^2(z-z_j)\left(k_{j\mu}+\rho_j\frac{\partial}{\partial z}+\bar{\rho}_j\frac{\partial}{\partial\bar{z}}\right)X^\mu(z,\bar{z})\right)\right]_{\rho_j=0}\end{aligned}$$

and so, writing

$$J_\mu(z) = -i\sum_{j=1}^n \delta^2(z-z_j)\left(k_{j\mu}+\rho_j\frac{\partial}{\partial z}+\bar{\rho}_j\frac{\partial}{\partial\bar{z}}\right)$$

the amplitude of n massless states may be written as

$$\mathcal{A}_n = (-1)^n g_c^{n-2}\prod_{i=1}^n \varepsilon_{\mu\nu}\left[\frac{\partial^2}{\partial\rho_j\partial\bar{\rho}_j}\exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2\omega\ J(z)J(\omega)G(z,\omega)\right)\right]_{\rho_j\bar{\rho}_j=0}$$

where the ordering in the exponent matters as $J(z)$ is now a differential operator on the worldsheet.

It can be helpful to split the Green's function into $z,\bar{z}$ dependent parts

$$G(z,\omega) = -\frac{\alpha'}{2}\ln(z-\omega) - \frac{\alpha'}{2}\ln(\bar{z}-\bar{\omega}).$$

The above expression then factorises

$$\begin{aligned}\mathcal{A}_n &\sim (-1)^n g_c^{n-2}\prod_{i=1}^n \varepsilon_{\mu\nu}\left[\frac{\partial^2}{\partial\rho_j\partial\bar{\rho}_j}\exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2\omega\ J(z)J(\omega)G(z,\omega)\right)\right]_{\rho_j\bar{\rho}_j=0}\\ &= (-1)^n g_c^{n-2}\prod_{i=1}^n \varepsilon_{\mu\nu}\left[\frac{\partial}{\partial\rho_j}\exp\left(-\frac{\alpha'}{4}\int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2\omega\ j(z)j(\omega)\ln(z-\omega)\right)\right]_{\rho_j=0}\\ &\quad\times\left[\frac{\partial}{\partial\bar{\rho}_j}\exp\left(-\frac{\alpha'}{4}\int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2\omega\ \bar{j}(z)\bar{j}(\omega)\ln(\bar{z}-\bar{\omega})\right)\right]_{\bar{\rho}_j=0}\end{aligned} \tag{23.2}$$

where

$$j(z) = -i\sum_{j=1}^n \delta^2(z-z_j)\left(\frac{1}{2}k_{j\mu}+\rho_j\frac{\partial}{\partial z}\right)$$

If we write $\tilde{X}^\mu(z,\bar{z}) = \tilde{X}^\mu(z) + \bar{X}^\mu(\bar{z})$, the amplitude may be written as

$$\mathcal{A}_3 = g_c^{n-2}(2\pi)^{26}|(z_1-z_2)(z_2-z_3)(z_3-z_1)|^2\,\delta^{26}\left(\sum_{i=1}^{3} k_i^\mu\right)$$
$$\times \int \prod_{j=4}^{n} \mathrm{d}^2 z_j \prod_{j=1}^{n} \varepsilon_{\mu_j\nu_j} \left\langle \prod_{j=1}^{n} \partial\tilde{X}^{\nu_j} e^{ik_j\cdot\tilde{X}(z_j)} \right\rangle \left\langle \prod_{j=1}^{n} \bar{\partial}\bar{X}^{\mu_j} e^{ik_j\cdot\bar{X}(\bar{z}_j)} \right\rangle,$$

where the interesting part of the amplitude is then we note that

$$\left\langle \prod_{j=1}^{n} \partial X^{\mu_j} e^{ik_j\cdot X(z_j)} \right\rangle = \frac{1}{i^n}\left[\prod_{j=1}^{n} \frac{\partial^n}{\partial\rho_1 ... \partial\rho_n} \exp\left(-\frac{\alpha'}{4}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\omega\; j(z)j(\omega)\ln(z-\omega)\right)\right]_{\rho_j=0}$$

and its right-moving counterpart. Substituting the expressions for $j(z)$ into the exponent gives

$$\begin{aligned}
W[j] &:= \exp\left(-\frac{\alpha'}{4}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\omega\; j(z)j(\omega)\ln(z-\omega)\right) \\
&= \exp\left(\frac{\alpha'}{4}\sum_{i\neq j} \left(\frac{1}{2}k_{i\mu} + \rho_i\frac{\partial}{\partial z_i}\right)\left(\frac{1}{2}k_{j\mu} + \rho_j\frac{\partial}{\partial z_j}\right)\ln(z_i - z_j)\right) \\
&= \prod_{i<j} |z_i - z_j|^{\frac{\alpha'}{2}k_i\cdot k_j} \times \exp\left(\frac{\alpha'}{2}\sum_{i<j}\frac{\rho_i\cdot\rho_j}{(z_i-z_j)^2} + \frac{\alpha'}{2}\sum_{i\neq j}\frac{k_i\cdot\rho_j}{z_i - z_j}\right)
\end{aligned}$$

We then have

$$\left\langle \prod_{j=1}^{n} \partial X^{\mu_j} e^{ik_j\cdot X(z_j)} \right\rangle = \frac{1}{i^n}\left[\prod_{j=1}^{n} \frac{\partial^n W[j]}{\partial\rho_1 ... \partial\rho_n}\right]_{\rho_j=0}$$

23.2.1 The three-point graviton amplitude

We first note that

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2}(k_1+k_2)^2 = \frac{\alpha'}{2}k_3^2 = 0,$$

as all states are massless. Thus

$$\prod_{i<j} |z_i - z_j|^{\frac{\alpha'}{2}k_i\cdot k_j} = 1$$

With work, one finds that

$$\left\langle \prod_{j=1}^{3} \partial X^{\mu_j} e^{ik_j\cdot X(z_j)} \right\rangle = \left(\frac{\alpha'}{2}\right)^2 \frac{T^{\mu_1\mu_2\mu_3}}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}.$$

where

$$T^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2}k_2^{\mu_3} + \eta^{\mu_2\mu_3}k_3^{\mu_1} + \eta^{\mu_3\mu_1}k_1^{\mu_2} + \frac{\alpha'}{2}k_3^{\mu_1}k_1^{\mu_2}k_2^{\mu_3}$$

Putting this with the similar expression for the anti-holomrophic sector and reintroducing the zero modes and ghosts gives

$$\boxed{\mathcal{A}_3 = g_c(2\pi)^{26}\delta^{26}\left(\sum_{i=1}^{3} k_i^\mu\right)\varepsilon_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2}\varepsilon_{\mu_3\nu_3}T^{\mu_1\mu_2\mu_3}T^{\nu_1\nu_2\nu_3}}$$

A few comments are in order.

- The terms quadratic in momenta are consistent with that given by perturbation of the Einstein-Hilbert action (with a suitable identification of the gravitational coupling constant).
- The quartic and sextic momenta come from higher derivative corrections to the Einstein-Hilbert action. These enter at order α' and vanish in the field theory limit where the string sclae is small compared to the curvature scale on the background.

24 Loops and Beyond

A brief overview of some more advanced topics are given.

24.1 One Loop

This is off the syllabus but it is interesting to see how the previous arguments at tree level are modified at one-loop. The major change here is that the moduli space is not trivial. In the one-loop case the moduli space is one-complex dimensional, the modulus, given by the complex structure of the torus

$$\tau = \tau_1 + i\tau_2.$$

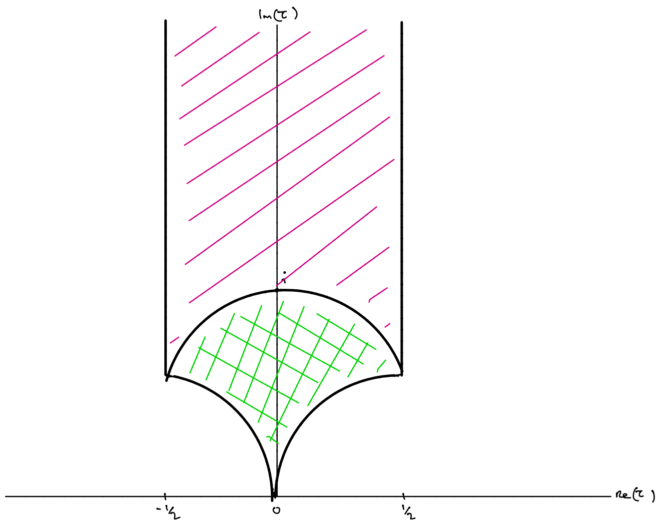
As discussed earlier, the moduli space is given by the upper half plane modded out by the action of the modular group. There are many choices but, by convention, we shall choose the fundamental domain that we integrate over to be

$$\mathcal{M}_1 = \left\{\tau = \tau_1 + i\tau_2 \middle| \tau_2 > 0; -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, |\tau| \geq 1\right\}.$$

This region is the purple shaded area in the picture below. The green shaded area gives an alternative choice.

Another change is that the conformal Killing group is $U(1) \times U(1)$, which is finite-dimensional, thus we can just divide out by the volume of this group. Since the CKVs are just translations of the torus, $|U(1) \times U(1)|$ is just the area of the torus, the area of the torus is

$$|U(1) \times U(1)| = \tau_2^2,$$



and it is simpler just to integrate over all punctures and divide out by the volume of the conformal Killing group explicitly.

The n-point amplitude then takes the form

$$\mathcal{A}_n = \frac{1}{|U(1)\times U(1)|}\int_{\mathcal{M}_1}\frac{\mathrm{d}^2\tau}{\tau_2^2}\int \mathcal{D}b\mathcal{D}c\mathcal{D}\bar{b}\mathcal{D}\bar{c}\;(\mu_\tau|b)(\bar{\mu}_\tau|\bar{b})\;c\bar{c}\,e^{-S[b,c]}\langle V_1 \ldots V_n\rangle_X$$

24.1.1 The ghost sector

We focus on the ghost sector first. Recall that $(\mu_\tau|b) = (\partial_\tau h_{ab}|b^{ab})$. How does the metric depend on the modulus τ? The line element on the worldsheet is given by $\mathrm{d}s^2 = \mathrm{d}z\,\mathrm{d}\bar{z}$. Consider a non-conformal deformation of the metric

$$h_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \to \begin{pmatrix} \bar{\varepsilon} & \frac{1}{2} \\ \frac{1}{2} & \varepsilon \end{pmatrix}.$$

Then

$$\mathrm{d}s^2 \to (1+\varepsilon+\bar{\varepsilon})\,\mathrm{d}z'\,\mathrm{d}\bar{z}',$$

where $z' = z + \varepsilon(z+\bar{z}) + \mathcal{O}(\varepsilon^2)$. The factor in front of the metric is simply a Weyl factor. Under this change, the periodicity of the coordinates changes as

$$z \sim z+\tau \qquad \Longrightarrow \; z' \sim z'+\tau'$$

where $\tau' = \tau - 2i\tau_2\varepsilon + \mathcal{O}(\varepsilon^2)$. Thus

$$\delta h_{\bar{z}\bar{z}} = \varepsilon \qquad \leftrightarrow \delta\tau = -2i\tau_2\varepsilon,$$

and so

$$\partial_\tau h_{\bar{z}\bar{z}} = \frac{i}{2\tau_2},$$

giving

$$(\mu_\tau|b) = \frac{i}{2\tau_2}\int_\Sigma \mathrm{d}^2 z\, b(z).$$

Thus the net ghost correlation function is $\langle b\bar{b}c\bar{c}\rangle$ which gives a function of τ_2.

24.1.2 The matter setor

More significant are the changes to the matter calculation. The manipulations that lead to the contributions

$$\langle V_1 ... V_n \rangle_X \sim \exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma} d^2z\, d^2\omega\ J^\mu(z) G(z,\omega) J_\mu(\omega)\right)(2\pi)^{26}\delta^{26}\left(\sum_{i=1}^n k_i^\mu\right)$$

are unchanged. The Green's function is required to have the correct periodicity under $z \sim z+\tau$ and so will depend explicitly on the complex structure τ. The correct Green's function is

$$G(z,\omega;\tau) = -\frac{\alpha'}{2}\ln F_\tau(z,\omega)$$

where

$$F_\tau(z,\omega) = \exp\left(\frac{\pi(z-\bar{z}-\omega-\bar{\omega})^2}{2\tau_2}\right)\left|\frac{\Theta_1(z-\omega;\tau)}{\Theta_1'(0;\tau)}\right|^2$$

where the first factor ensures that the function is single valued on the torus and the second involves the celebrated *theta functions*[34] The amplitude takes the form

$$A_n \sim \delta^{26}\left(\sum_{i=1}^n k_i^\mu\right)\int d^2z_1 ... d^2z_n \prod_{i<j} F_\tau(z,\omega)^{\alpha' k_i\cdot k_j}.$$

A few comments are in order.

- One can perform a similar calculation of the one-loop amplitude from the first quantised perspective in field theory. There one is integrating over world lines and one finds a similar integral over the modulus ℓ which parameterises the length of the circle in the loop amplitude. This is the field theory analogue of the complex structure modulus considered here. In the field theory calculations, there is a UV divergence as $\ell \to 0$ in the integral. The analogous region $\tau = 0$ is absent in string theory. Thus the UV finiteness of the theory at a particular order of perturbation theory is guaranteed by modular invariance.

- There are no UV divergences in bosonic string theory; however, there is an IR divergence as $\tau_2 \to \infty$. This corresponds to a long propagation time for the Tachyon and is a problem for the bosonic string. Matters are better in the superstring as there is no Tachyon in the spectrum.

- Higher loop calculations are hard. This is mostly down to the complicated moduli spaces of higher genus Riemann surfaces and the difficulty of performing the integrals explicitly.

[34] The $\Theta_1'(0;\tau)$ comes from subtracting off diverges (normalised self-contractions).

24.2 A reevaluation of moduli space

Scattering amplitudes are computed by inserting a vertex operator into a puncture of a punctured Riemann surface of genus g. We denote such Riemann surfaces, with n punctures by $\Sigma_{n,g}$. Punctured Riemann surfaces have moduli associated with the punctures. Clearly the location of the puncture may be specified by the coordinates σ^a of the puncture, which gives an additional $2n$ moduli corresponding to the two coordinates for each puncture. The real dimension of the moduli space is then $s+2n$. We can then write

$$\int_{\mathcal{M}_g} \mathrm{d}^s t \to \int_{\mathcal{M}_{g,n}} \mathrm{d}^{s+2n} t = \int_{\mathcal{M}_g} \mathrm{d}^s t \int \prod_{i=1}^{n} \mathrm{d}^2 z_i$$

At tree level, this means

$$\begin{aligned} \mathcal{A}_n &= \int \prod_{i=4}^{n} \mathrm{d}^2 z_i \left\langle \phi(z_1)\phi(z_2)...\phi(z_n) \right\rangle_X \\ &= \int_{\mathcal{M}_{0,n}} \mathrm{d}^{2n-6} z_i \left\langle \phi(z_1)\phi(z_2)...\phi(z_n) \right\rangle_X \end{aligned}$$

If we are now taking the dimension of the moduli space to be $2n-6$, what of the $(\mu_I|b)$ insertions? We find that they are precisely what we need to eliminate the $c\bar{c}$ ghost insertions in the unintegrated vertex operators $U(z,\bar{z})$, so we may write the amplitude as

$$\mathcal{A}_n = \int_{\mathcal{M}_{0,n}} \mathrm{d}^{2n-6} z_i \prod_{i=1}^{n-3} (\mu_i|b)(\bar{\mu}_i|\bar{b}) \left\langle U(z_1)U(z_2)...U(z_n) \right\rangle$$

The b-ghost insertions remove the c-ghosts from $n-3$ of the vertex operators. The c-ghosts from the remaining three vertex operators give a factor of

$$|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^2$$

which we are free to fix. This perspective can be useful when studying degenerating Riemann surfaces that describe the situation when punctures collide.

Part VI

Open Strings and D-Branes

In this final section we consider worldsheets with boundary, i.e. *open strings*. In some sense it is a little odd to put this topic as the last in the course as, historically, it was open strings and their connection to non-abelian gauge theory which were stumbled across first. However, we shall see that the inclusion of boundaries on the worldsheet requires only a relatively minor modification of the techniques we have discussed in previous sections for the closed string (although the physical differences are profound) and we will be able to mount a relatively brisk investigation of open strings.

25 Open String Theory

25.1 Neumann and Dirichlet Boundary Conditions

25.2 Quantization

26 D-Branes

26.1 The Dirac-Born-Infeld Action

27 Chan-Paton factors and gauge symmetry

28 The Spacetime Perspective

29 Scattering Amplitudes

29.1 Vertex Operators

29.2 The Ubiquity of Gravitation

Part VII

Appendices

A Grassmann Integration

A short note on Grassmann integration will be added. In the meantime consult standard Quantum Field Theory Textbooks such as Ryder or Peskin and Schroeder.

B The Ghost Propagator and the Path Integral

We introduce anti-commuting sources for the ghosts

$$S_{J,K}[b,c] = \frac{1}{2\pi}\int_\Sigma b\bar{\partial}c + \int_\Sigma Jc + bK$$

It will be useful to define the shifted fields

$$B(z) \equiv b(z) + \int_\Sigma \mathrm{d}\omega S(z,\omega)J(\omega), \qquad C(z) \equiv c(z) + \int_\Sigma \mathrm{d}\omega S(z,\omega)K(\omega)$$

where $S(z,\omega)$ is the classical Green's function for $\bar{\partial}$ satisfying

$$\bar{\partial}S(z,\omega) = 2\pi\delta^2(z-\omega).$$

It is not too hard to show[35]

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2z B\bar{\partial}C = \frac{1}{2\pi}\int_\Sigma b\bar{\partial}c + \int_\Sigma Jc + bK + \int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2\omega J(z)S(z,\omega)K(\omega).$$

The relationship between (b,c) and (B,C) is linear so there is no additional Jacobian if we change integration variables in the functional integral. The generating functional becomes

$$Z[J,K] = \Big(\int \mathcal{D}B\mathcal{D}C\; e^{-\frac{1}{2\pi}\int_\Sigma B\bar{\partial}C}\Big) \times \exp\Big(-\int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2\omega J(z)S(z,\omega)K(\omega)\Big)$$

The term in the first brackets is just $Z[0,0]$ so we should normalise by diving out by it, leaving

$$Z[J,K] = \exp\Big(-\int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2\omega J(z)S(z,\omega)K(\omega)\Big).$$

The two-point function is given by

$$\langle b(z)c(\omega)\rangle = \frac{\delta^2 Z[J,K]}{\delta J(z)\delta K(\omega)} = S(z,\omega),$$

[35] A term must be integrated by parts and the surface term discarded; otherwise, the calculation is straightforward.

so we see that the two point function is simply the classical Green's function. This far the discussion would hold for any genus Riemann surface. To solve for the green's function we need to select a genus of interest so that the Green's function has the appropriate periodicity conditions. To find the Green's function, we make use of the result on the sphere that

$$\bar{\partial}\left(\frac{1}{z-\omega}\right) = 2\pi\delta^2(z-\omega),$$

to identify[36]

$$S(z,\omega) = \frac{1}{z-\omega}.$$

We then have that

$$\boxed{\langle b(z)c(\omega)\rangle = \contraction{}{b(z)}{}{c}b(z)c(\omega) = \frac{1}{z-\omega}}$$

[36]There are no globally holomorphic zero modes of $\bar{\partial}$ on the sphere. This changes at higher genus.