String Theory: Example Sheet 1

Corrections and suggestions should be emailed to R.A.Reid-Edwards@damtp.cam.ac.uk.

1. Show that the Nambu-Goto and Polyakov expressions for the relativistic string are classically equivalent. The Polyakov action has reparameterisation *and* Weyl invariance, yet the Nambu-Goto action only has reparametrisation invariance. What is going on?

2. (a) h_{ab} is a two-dimensional worldsheet metric and $h = \det h_{ab}$. Using

$$\delta h = h\, h^{ab}\delta h_{ab}, \qquad \delta h^{ab} = -h^{ac}h^{bd}\delta h_{cd},$$

derive an expression for the energy-momentum tensor

$$T_{ab} = \frac{4\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{ab}}$$

from the Polyakov action for the string.

(b) Now choose a metric such that the line element on the worldsheet is $-\,\mathrm{d}\tau^2 + \mathrm{d}\sigma^2$. Show that, in worldsheet light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, T_{ab} may be written as

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X^\nu \eta_{\mu\nu}, \qquad T_{--} = -\frac{1}{\alpha'}\partial_- X^\mu \partial_- X^\nu \eta_{\mu\nu}, \qquad T_{+-} = 0$$

Hence show that

$$\partial_- T_{++} = 0 = \partial_+ T_{--}.$$

(c) Find an expression for T_{++} and T_{--} in terms of the oscillator modes α^μ_n and $\bar{\alpha}^\mu_n$ at $\tau = 0$, defined in the lectures and hence find an expression for the Virasoro modes ℓ_n, where

$$T_{--}(\sigma) = -\sum_n \ell_n\, e^{in\sigma}, \qquad T_{++}(\sigma) = -\sum_n \bar{\ell}_n\, e^{-in\sigma}.$$

3. (a) Using the Poisson bracket relation

$$\{\alpha^\mu_m, \alpha^\nu_n\} = -im\eta^{\mu\nu}\delta_{m+n,0},$$

show that

$$\{\ell_m, \alpha^\mu_n\} = in\alpha^\mu_{m+n}.$$

(b) Hence show that

$$\{\ell_m, \ell_n\} = -i(m-n)\ell_{m+n}.$$

4.* In this question, we consider the worldsheet to be an infinite cylinder with a Euclidean metric $\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}\sigma^2$. One might like to think of this as a Wick rotation $\tau \to i\tau$ of the flat Minkowski metric. We map worldsheet to the complex plane using

$$z = e^{\tau - i\sigma}, \qquad \bar{z} = e^{\tau + i\sigma}.$$

(a) Show that this diffeomorphism is also a Weyl transformation.

(b) Explain why one can think of the worldsheet now as a sphere with two points removed.

(c) Write the mode expansion of X^μ as a function of $(z, \bar{z})$ and show that we can write

$$X^\mu(z, \bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z}).$$

(d) Assuming $|z| < |\omega|$ (i.e. $\tau_z < \tau_\omega$), and using the canonical commutation relations for the oscillator modes α_n^μ, show that the two-point function for the '$\bar{z}$-independent' part of the embedding fields is

$$\langle 0|\partial_z X^\mu(z)\partial_\omega X^\nu(\omega)|0\rangle = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}$$

The result

$$\sum_{n>0} n y^{n-1} = \frac{1}{(1-y)^2}, \qquad |y| < 1.$$

may be useful. By directly integrating the above result, find a similar expression for

$$\langle 0|X^\mu(z)X^\nu(\omega)|0\rangle.$$

5. Consider the Polykov action, but with boundary conditions at $\sigma = 0, \pi$. This describes the open string.

(a) Show that the equations of motion require the boundary conditions to be either

$$\delta X^\mu = 0, \qquad \text{or} \qquad \partial_\sigma X^\mu = 0$$

on the boundary. The first are called Dirichlet boundary conditions and signal the presence of D-branes. The second are called Neumann boundary conditions.

(b) Show that the boundary conditions imply only one set of independent oscillators.

(c) Impose canonical commutation relations on the modes and show that the theory has a tachyon in the ground state.

(d) Show that

$$|A\rangle = A_\mu(k)\alpha_{-1}^\mu|k\rangle,$$

is massless (what do you have to assume about L_0 for this to be the case?). Show that $k^\mu A_\mu = 0$ and that the state

$$|\lambda\rangle = \lambda(k)k_\mu\alpha_{-1}^\mu|k\rangle,$$

is spurious (i.e. $\langle\lambda|\lambda\rangle = 0$). Give a spacetime interpretation to the Fourier transform of $A_\mu(k)$.

6.* Given the canonical commutation relations, one can show that the Virasoro generators L_n in the quantum theory do not satisfy the Witt algebra, but instead satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0},$$

where A is the anomaly.

(a) Show that the $SL(2)$ sub-algebra generated by $L_0, L_{\pm 1}$ has no anomaly.

(b) Assume that A has the form

$$A(m) = Bm^3 + Cm,$$

where B and C are constants. Show, by considering $\langle 0|[L_m, L_{-n}]|0\rangle$ for suitable values of m and n, that

$$A(m) = \frac{D}{12}m(m^2-1),$$

where D is the dimension of spacetime. Does this mean the quantum string is only consistent in $D = 0$ dimensions? What have we missed?

7. Show that

$$S[\phi] = \langle\phi|(L_0^+ - 2)|\phi\rangle,$$

where

$$|\phi\rangle = \int \mathrm{d}k\, \widetilde{\phi}(k)|k\rangle,$$

gives the standard Klein-Gordon action for a Tachyon ϕ. Note, you will need the zero-mode Fourier transform relation

$$\widetilde{\phi}(k) = \frac{1}{2\pi}\int \mathrm{d}x\, \phi(x)e^{ik\cdot x}.$$

This isn't quite right (what have we missed?) but gives a rough idea of how you might try to go off-shell to find a second quantised theory of strings.

String Theory: Example Sheet 2

Corrections and suggestions should be emailed to R.A.Reid-Edwards@damtp.cam.ac.uk.

1. An inner product of two worldsheet tensors $A_{a_1...a_n}$ and $B_{b_1...b_n}$ is defined as

$$(A|B) \equiv \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{a_1 b_1} ... h^{a_n b_n} A_{a_1...a_n} B_{b_1...b_n}.$$

By requiring the variation of the metric in moduli space $\delta_t h_{ab}$ to be orthogonal to the variation given by Diffeomorphisms and Weyl transformations, show that $\delta_t h_{ab}$ must satisfy

$$h^{ab}\delta_t h_{ab} = 0, \qquad (\mathcal{P}^T \delta_t h)_a = 0,$$

where $\mathcal{P}^T$ is the conjugate to $\mathcal{P}$ under this inner product; i.e. $(A|\mathcal{P}B) = (\mathcal{P}^T A|B)$ and $\mathcal{P}$ is that operator which generates traceless diffeomorphisms in the general variation of the metric

$$\delta h_{ab} = (\mathcal{P}v)_{ab} + (2\omega - \partial_c v^c)h_{ab} + \delta_t h_{ab}.$$

Thus we see that the conformal Killling vectors are in Ker($\mathcal{P}$), whilst the moduli variations of the metric are in are in Ker($\mathcal{P}^T$).

2. Let

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_h = \int \mathcal{D}X \mathcal{D}h \, e^{iS[X,h]} \, \mathcal{O}_1 ... \mathcal{O}_n,$$

where $S[X,h]$ is the Polyakov action, be the correlation function for some observables $\mathcal{O}_i$ calculated using a worldsheet with metric h_{ab}. Starting with this path integral expression for the correlation function show that, under a change in the worldsheet metric, the first order change in the correlaton function is given by

$$\delta_h \langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_h = -\frac{1}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \, \sqrt{-h} \delta h_{ab} \langle T^{ab}(\sigma) \mathcal{O}_1 ... \mathcal{O}_n \rangle_h.$$

By considering the change of the correlation function under a Weyl transformation, show that

$$\langle T^a_a(\sigma) \mathcal{O}_1 ... \mathcal{O}_n \rangle_h = 0.$$

3. Consider the Polyakov action $S[X]$ with *fixed* worldsheet metric h_{ab} describing a closed string embedded into flat Minkowski space. Show that, under the transformation $\delta_v X^\mu = v^a \partial_a X^\mu$, the action changes as

$$\delta S[X] = \frac{1}{2\pi} \int_\Sigma \mathrm{d}^2\sigma \, (\partial^a v^b) T_{ab},$$

where T_{ab} is the stress tensor.

4. Consider a d-dimensional flat spacetime with Minkowski metric $\eta_{\mu\nu}$ and coordinates x^μ. An infinitesimal diffeomorphism is given by

$$x^\mu \to f^\mu(x) = x^\mu + \epsilon v^\mu(x) + \ldots$$

where $\epsilon \ll 1$ is a small dimensionless constant.

(a) Show that $v^\mu(x)$ is a conformal Killing vector if, to leading order in ϵ

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{d}\eta_{\mu\nu}\partial_\lambda v^\lambda = 0.$$

(b) By considering a further derivative of this equation, show that,

$$\Big(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu\Big)\partial_\lambda v^\lambda = 0,$$

and therefore

$$(d-1)\Box\partial_\lambda v^\lambda = 0.$$

(c) Show that the final equation derived in part (b) implies that, in $d > 2$, $v(x)^\mu$ is of the form

$$v_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\lambda}x^\nu x^\lambda$$

where a_μ, $b_{\mu\nu}$, and $c_{\mu\nu\lambda}$ are constants.

The a_μ parameterise translations. We can decompose into symmetric and antisymmetric parts: $b_{\mu\nu} = \lambda\eta_{\mu\nu} + m_{\mu\nu}$, where λ parameterises dilations and $m_{\mu\nu}$ are the Lorentz transformations (or rotations in d-dimensions). By further analysis one can show that the important information in $c_{\mu\nu\lambda}$ can be encoded in a vector $b_\mu = \frac{1}{d}c^\nu{}_{\nu\mu}$ and the b_μ parameterise special conformal transformations $x^\mu \to x^\mu + 2(x\cdot b)x^\mu - (x\cdot x)b^\mu$.

5. z is a coordinate on the complex plane. Consider the vector fields

$$\ell_n = -z^{n+1}\partial_z, \qquad n \in \mathbb{Z}.$$

(a) Show that the ℓ_n satisfy the Witt algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}.$$

(b) Using the change of variable, $z = -\omega^{-1}$, show that the only ℓ_n holomorphic at $\omega = 0$ and $z = 0$ are ℓ_0 and $\ell_{\pm 1}$. Show that these generate $SL(2)$. Comment on which conformal transformations are globally defined on the Riemann sphere.

6. The weight $(2,0)$ component of the stress tensor may be written as the mode expansion

$$T(z) = \sum_n L_n z^{-n-2}.$$

(a) Show that

$$L_n = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i}\, z^{n+1}\, T(z).$$

(b) Let $\Phi(z,\bar{z})$ be a primary field of weight $(1,1)$ with mode expansion

$$\Phi(z,\bar{z}) = \sum_{n,\bar{n}} \phi_{n\bar{n}} z^{-n-1}\bar{z}^{-\bar{n}-1},$$

where you may assume

$$[L_m, \phi_{n\bar{n}}] = -n\phi_{n+m,\bar{n}}, \qquad [\bar{L}_{\bar{m}}, \phi_{n\bar{n}}] = -\bar{n}\phi_{n,\bar{n}+\bar{m}}.$$

By considering the commutator of $\Phi(z,\bar{z})$ with the charge

$$Q = \oint_{\omega=0} \frac{\mathrm{d}\omega}{2\pi i}\, v(\omega)T(\omega) + \oint_{\bar{\omega}=0} \frac{\mathrm{d}\bar{\omega}}{2\pi i}\, \bar{v}(\bar{\omega})\bar{T}(\bar{\omega})$$

for appropriately chosen $v(\omega)$, show that

i. The action of translations $z \to z + c$ on the field $j^\mu(z)$, where $c \in \mathbb{C}$ are generated by L_{-1}.
ii. Dilations $z \to e^\lambda z \approx z + \lambda z$, where $\lambda \in \mathbb{R}$ are generated by L_0^+.
iii. Rotations $z \to e^{i\theta} z \approx z + i\theta z$, where $\theta \in \mathbb{R}$ are generated by L_0^-.

Hint: Start by choosing appropriate forms for $v(z)$ in each case (e.g. $v(z) = \lambda z$ in the case of dilations).

7. * In this question, you may assume the OPE

$$\partial X^\mu(z)\partial X^\nu(\omega) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2} + \ldots$$

(a) Given

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu z^{-n-1}$$

show that

$$\alpha_n^\mu = i\sqrt{\frac{2}{\alpha'}}\oint \frac{\mathrm{d}z}{2\pi i}\, z^n\, \partial X^\mu(z)$$

and therefore show that

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{m+n,0}.$$

(b) By finding the OPE of $X^\mu(z)$ with $T(\omega)$, prove that $\partial X^\mu(z)$ has conformal weight $(h,\bar{h}) = (1,0)$.

(c) Show that $\partial^n X^\mu(z)$ has conformal weight $h = n$ but is not a primary operator for $n > 1$.

8. * Let $W(\omega)$ be the chiral operator

$$W(\omega) := \varepsilon_\mu : \partial X^\mu(\omega)\, e^{ik\cdot X(\omega)} :$$

where k_μ and ε_μ are arbitrary constant spacetime vectors. By considering the OPE with the stress tensor $T(z)$, show that $V(\omega)$ has weight

$$h = 1 + \frac{\alpha' k^2}{4}.$$

Explain why $V(\omega)$ is a primary field if $k \cdot \varepsilon = 0$.

Without doing any further calculations, give conditions on $\varepsilon_{\mu\nu}$ and k_μ for

$$V(\omega, \bar{\omega}) := \varepsilon_{\mu\nu} : \partial X^\mu(\omega) \bar{\partial} X^\nu(\bar{\omega}) \, e^{ik \cdot X(\omega, \bar{\omega})} :$$

to be a primary field of weight $(1, 1)$.

String Theory: Example Sheet 3

Corrections and suggestions should be emailed to R.A.Reid-Edwards@damtp.cam.ac.uk.

1. By considering a general infinitesimal mobius transformation, show that the infinitesimal action of $SL(2;\mathbb{C})$ on a complex coordinate z is of the form

$$z \to z' = \alpha + \beta z + \gamma z^2,$$

where α, β and γ are constants.

Hence show that the volume element of $SL(2;\mathbb{C})$ may be written in terms of three points on the Riemann sphere with coordinates z_1, z_2 and z_3 as

$$\mathrm{dVol}(SL(2;\mathbb{C})) = \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2}.$$

2. Using the mode expansion

$$c(z) = \sum_n c_n z^{-n+1},$$

where $c_n|0\rangle = 0$ for $n > 1$, show that

$$\langle c(z_1)c(z_2)c(z_3)\rangle = K(z_1 - z_2)(z_2 - z_3)(z_3 - z_1),$$

and find the constant K.

Using this result and that for question 1 show, for an appropriate choice of puncture labelling, that we may write

$$\prod_{i=1}^{n} \mathrm{d}^2 z_i \left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle \Big/ \mathrm{dVol}(SL(2;\mathbb{C})) = \prod_{i=4}^{n} \mathrm{d}^2 z_i.$$

3.* Show that the OPE of two operators of the form $: e^{ik\cdot X} :$ takes the form

$$: e^{ik_1\cdot X(z_1)} :: e^{ik_2\cdot X(z_2)} := |z_1 - z_2|^{\alpha' k_1\cdot k_2} : e^{ik_1\cdot X(z_1)} e^{ik_2\cdot X(z_2)} : .$$

You may find it useful to write the exponential operator as

$$: e^{ik\cdot X(z)} := \sum_{n=0}^{\infty} \frac{i^n}{n!} k_{\mu_1} ... k_{\mu_n} : X^{\mu_1}(z) ... X^{\mu_n}(z) :$$

and consider $X^\mu(z_1) : e^{ik\cdot X(z_2)} :$ first.

4. Given the point particle action

$$S[X,e] = \int \mathrm{d}\tau \left(\frac{1}{2} e^{-1} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - e\dot{b}c + iB(e-1) \right),$$

(a) Show that this action is invariant under the BRST transformation

$$\delta_Q X^\mu = i\varepsilon c\dot{X}^\mu, \qquad \delta_Q e = i\varepsilon\frac{\mathrm{d}}{\mathrm{d}\tau}(ce), \qquad \delta_Q b = \varepsilon B, \qquad \delta_Q c = i\varepsilon c\dot{c}.$$

(b) Integrate out B. Show that the action becomes

$$S[X,e] = \int \mathrm{d}\tau \left(\frac{1}{2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} - \dot{b}c\right),$$

and find the BRST transformations that leave this action invariant.

(c) Find the BRST operator for the gauge-fixed action and show that it generates the correct BRST transformations up to terms which vanish on the equations of motion. You may assume the canonical commutation relations

$$[P_\mu, X^\nu] = -i\delta^\nu_\mu, \qquad \{b,c\} = 1.$$

5. Let

$$j_B(z) = c(z)\left(T_X(z) + \frac{1}{2}T_{\mathrm{gh}}(z)\right) + \kappa\partial^2 c(z),$$

be the the BRST current. By requiring $j_B(z)$ to transform as a weight $(1,0)$ conformal field, find a suitable value for the constant κ.

6. The stress tensor for the ghost sector is given by

$$T_{\mathrm{gh}}(z) =: (\partial b)c(z) : -2\partial(: bc(z) :).$$

Derive the $T_{\mathrm{gh}}(z)c(w)$ OPE and hence show that $c(w)$ is a conformal field of weight $h = -1$.

7.*

(a) State under what conditions is the operator

$$\varepsilon_{\mu\nu}c\bar{c}\partial X^\mu\bar{\partial}X^\nu e^{ik\cdot X} \tag{1}$$

BRST closed. (You may refer to results from the previous example sheet).

(b) Consider the operator

$$W = \lambda_\mu c(z)\partial X^\mu(z)\, e^{ik\cdot X(z,\bar{z})}.$$

Show that$\{\mathcal{Q}_B, W\}$ is of the form (1) where $\mathcal{Q}_B$ is the BRST operator and you should give $\varepsilon_{\mu\nu}$ in terms of λ_μ and k_μ. How does this illustrate the relationship between BRST-invariance and gauge-invariance in spacetime?

String Theory: Example Sheet 4

Corrections and suggestions should be emailed to R.A.Reid-Edwards@damtp.cam.ac.uk.

1. * Explaining your reasoning carefully, show that

$$\lim_{z,\bar{z}\to 0} -\frac{2}{\alpha'}\varepsilon_{\mu\nu} : c(z)\bar{c}(\bar{z})\partial X^{\mu}(z)\bar{\partial}X^{\nu}(\bar{z})e^{ik\cdot X(z,\bar{z})} : |0\rangle = \varepsilon_{\mu\nu}c_1\bar{c}_1\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle,$$

and explain how this relationship between operators and states is useful in describing string scattering amplitudes.

2. Compute the scattering amplitude for three tachyons in a flat spacetime using path integral methods as discussed in lectures. Derive the same result using the OPE given in question 3 of example sheet 3.

3. Using path integral methods, show that the four-point closed string tachyon tree-level scattering amplitude is

$$\mathcal{A}_4 = g_c^2\,\mathcal{C}\,\delta^{26}\left(\sum_{i=1}^{4} k_i^{\mu}\right)\int \mathrm{d}^2 z\,|z|^{\alpha' k_1\cdot k_4}|1-z|^{\alpha' k_2\cdot k_4},$$

where $\mathcal{C}$ is a constant that you need not determine. You may find it helpful to choose the points

$$z_1 = 0, \qquad z_2 = 1, \qquad z_3 \to \infty, \qquad z_4 := z.$$

4. Using the OPE given in question 3 of example sheet 3 and the OPE of $\partial X^{\mu}(z)$ wtih $: e^{ik\cdot X(\omega)} :$, which you should derive, evaluate

$$\langle : \partial X^{\mu}(z_1)e^{ik_1\cdot X(z_1)} :: e^{ik_2\cdot X(z_2)} :: e^{ik_3\cdot X(z_3)} :\rangle.$$

Hence compute the scattering amplitude for a massless field of polarisation $\varepsilon_{\mu\nu}$ and momentum k_1 with two tachyon fields, with momenta k_2 and k_3 and show that it may be written in the form

$$\mathcal{A}_n = K g_c\,\varepsilon_{\mu\nu}\,k_{23}^{\mu}k_{23}^{\nu}\,\delta^{26}\left(\sum_{i=1}^{3} k_{i\mu}\right),$$

where K is a constant that you need not determine and $k_{23}^{\mu} = k_2^{\mu} - k_3^{\mu}$.

5. (a) By introducing the dummy (z and $\bar{z}$ independent) variables $\rho_{i\mu}$ show that

$$i\partial X^{\mu}(z_j)\,e^{ik_j\cdot X(z_j)} = \left[\frac{\partial}{\partial\rho_{j\mu}}\exp\left(i\sum_{j=1}^{n}\int_{\Sigma}\mathrm{d}^2 z\left(k_{j\mu} + \rho_{j\nu}\frac{\partial}{\partial z}\right)X^{\nu}(z)\delta^2(z-z_j)\right)\right]_{\rho_j=0}.$$

(b) Hence show that, up to terms which vanish upon contraction with physical polarisation tensors,

$$\left\langle \prod_{j=1}^{3} \partial X^{\mu_j} e^{ik_j\cdot X(z_j)} \right\rangle = \left(\frac{\alpha'}{2}\right)^2 \frac{T^{\mu_1\mu_2\mu_3}}{(z_1-z_2)(z_2-z_3)(z_3-z_1)},$$

where $k_j^2 = 0$ and

$$T^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2}k_2^{\mu_3} + \eta^{\mu_2\mu_3}k_3^{\mu_1} + \eta^{\mu_3\mu_1}k_1^{\mu_2} + \frac{\alpha'}{2}k_3^{\mu_1}k_1^{\mu_2}k_2^{\mu_3}.$$

(c) Hence write down the tree-level scattering amplitude for three gravitons (you do not need to perform further calculations and may quote the standard result for the ghost contribution at tree level).

(d) What is the physical significance of the $\alpha' \to 0$ limit. What is the significance of the α' contributions to this three point amplitude?

6. * Using path integral methods, find the three-point tree-level scattering amplitude for two tachyons and a massless state with polarisation $\varepsilon_{\mu\nu}$.

7. (a) A string embeds into a one-dimensional circular target space where the embedding coordinate X is subject to the identification $X \sim X + 2\pi R$ where R is the radius of the circle. Consider a deformation of the worldsheet theory given by adding the term

$$\Delta S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2z\, \varepsilon\, \partial X \bar{\partial} X,$$

to the action, where ε is a real constant. What is the target space interpretation of this deformation? Explain how this relates to vertex operators and describe the space of target spaces related by such deformations.

(b) Now consider a string embedding into a two-dimensional target space given by a square torus (i.e. complex structure $\tau = i$). Write down the vertex operators for the theory and explain their physical interpretation in terms of deformations of the background in which the string is embedded.

8. Write down the Polyakov action for a $n+1$ dimensional membrane. Is Weyl invariance a symmetry of the classical theory for any value of n?

The following questions go beyond the course and are included for interest. Do not attempt these questions at the expense of the earlier questions.

9. Consider a closed string with target space embedding into a spacetime with a circular direction of radius R

$$(X^i, Y) : \Sigma \to \mathbb{R}^{24,1} \times S^1,$$

so that the coordinate along the circle is periodic; $Y \sim Y + 2\pi R$.

(a) Explain why the periodicity condition on the string is

$$Y(\sigma + 2\pi, \tau) = Y(\sigma, \tau) + 2\pi R\omega$$

taking care to give an interpretation to $\omega \in \mathbb{Z}$ and hence find a mode expansion for $Y(\sigma, \tau)$.

(b) Show that the mass-shell and level-matching conditions are

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \bar{N} - 2), \qquad nw + N - \bar{N} = 0,$$

respectively where $n, w \in \mathbb{Z}$. N and $\bar{N}$ take their usual meaning. Hence show that the mass spectrum of the theory is invariant under the exchange of winding and momentum modes around the compact direction provided that R is exchanged with α'/R. What is the significance of this result?
Hint: Do not assume that $\alpha_0^\mu = \bar{\alpha}_0^\mu$.

(c) Consider the limit in which $R \to 0$, keeping α' fixed. How do the low energy degrees of freedom in the resulting 25 dimensional theory on $\mathbb{R}^{24,1}$ differ in field theory and string theory?

10. As in question 9, assume that the target space is $\mathbb{R}^{24,1} \times S^1$, where the radius of the circle is now taken to be $R = \sqrt{\alpha'}$.

(a) Using standard OPE results, find the singular terms in the OPEs of the operators

$$J^1(z) =: \sin\left(\frac{2}{\sqrt{\alpha'}}Y(z)\right) :, \qquad J^1(z) =: \cos\left(\frac{2}{\sqrt{\alpha'}}Y(z)\right) :, \qquad J^3(z) = \frac{i}{\sqrt{\alpha'}}\partial Y(z).$$

(b) Defining

$$J_n^a = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^n \, J^a(z),$$

show that the J_n^a satisfy

$$[J_m^a, J_n^b] = \frac{m}{2}\delta^{ab}\delta_{m+n,0} + i\varepsilon^{abc} J_{m+n}^c.$$

Such an algebra is called a Kac-Moody algebra and plays a crucial role in the construction of the Heterotic String theories.

11. For a string in a background B-field with field strength $H = \mathrm{d}B$ we may write the action as

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}h^{ab}g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu - \frac{1}{12\pi\alpha'}\int_{\mathcal{V}} \epsilon^{abc}H_{\mu\nu\lambda}(X)\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda$$

where $\mathcal{V}$ is a surface such that $\partial\mathcal{V} = \Sigma$.

(a) Given that H transforms under diffeomorphisms as $\delta_v H = i_v \, \mathrm{d}H + d(i_v H)$, show that the X^μ equations of motion are defined entirely by physics on the two-dimensional world-sheet Σ.

(b) Show that the theory does not depend on the form of the extension $\mathcal{V}$ of Σ.

Hint: Consider the difference between the actions given by two different extensions $\mathcal{V}$ and $\mathcal{V}'$, both with boundary Σ.