Chapter 1

Basic Concepts

1.1 Trajectories

We shall be concerned in this course with the motion of *particles*. Larger bodies will (with a few exceptions) be made up of collections of particles. We will find that large bodies can be represented as point particles for certain purposes (e.g., for the Earth orbiting the Sun), but that when rotations about an axis are involved we need to take account of the composition of the body.

We denote the trajectory of a particle by either $\mathbf{x}(t)$ or $\mathbf{r}(t)$ (sometimes interchangeably), a 3D vector function of time with components denoted either by (x_1, x_2, x_3) or (x, y, z). The vector velocity is $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ and the acceleration is $\ddot{\mathbf{x}}(t)$. Two particles with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ have a *relative displacement* of $\mathbf{r}_2 - \mathbf{r}_1$ (relative to the first particle) and a *relative velocity* of $\mathbf{v}_2 - \mathbf{v}_1$.

Note that many rules of calculus apply to vectors just as they do to scalars: for example, if $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are general vector functions of time then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a} \,.\, \mathbf{b}) = \dot{\mathbf{a}} \,.\, \mathbf{b} + \mathbf{a} \,.\, \dot{\mathbf{b}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a} \times \mathbf{b}) = \dot{\mathbf{a}} \times \mathbf{b} + \mathbf{a} \times \dot{\mathbf{b}}$$

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = (\dot{a}_1, \dot{a}_2, \dot{a}_3)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{a}\big(\lambda(t)\big) = \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}$$

$$\int_{t_1}^{t_2} \dot{\mathbf{a}}(t)\,\mathrm{d}t = \mathbf{a}(t_2) - \mathbf{a}(t_1).$$

These can be proved easily using suffix notation.

1.2 Units

To measure any physical, dimensional quantity we first need a *system of units*, which defines a fairly small set of *base units*. These base units can then be combined to produce a complete set of derived units for all possible physical quantities. The SI system is widely used, though there are many others (notably ones based on Imperial measures). In the SI system there are seven base units, viz., metre (m), kilogram (kg), second (s), ampere (A), Kelvin (K), mole (mol) and candela (cd) for measuring length, mass, time, electric current, temperature, "amount of substance" and light intensity respectively.

One metre (1 m) is the distance that light travels in a vacuum in $\frac{1}{299{,}792{,}458}$ s. (The speed of light in vacuum in m/s is therefore *defined*, rather than being something we could measure.)

One kilogram (1 kg) is the mass of a particular platinum–iridium cylinder held at Sèvres, near Paris.

One second (1 s) is defined to be the time taken for $9,192,631,770$ cycles of the radiation emitted by a Caesium-133 atom at rest at 0 K (resulting from transitions between the two hyperfine levels of its ground state).

One amp (1 A) is the current which, when passed through two perfect wires placed 1 m apart in a vacuum, produces a force of $2 \times 10^{-7}\,\mathrm{kg\,m/s^2}$ per metre of length.

Kelvin, mole and candela have similarly precise definitions.

All other SI units are formed from these base units. For instance, force is measured in Newtons (N), and $1\,\mathrm{N} \equiv 1\,\mathrm{kg\,m/s^2}$. Energy, to be introduced later, is measured in Joules ($1\,\mathrm{J} \equiv 1\,\mathrm{N\,m}$). Power, i.e., rate of production or loss of energy, is measured in Watts ($1\,\mathrm{W} \equiv 1\,\mathrm{J/s} \equiv 1\,\mathrm{kg\,m^2/s^3}$).

Electric charge is measured in Coulombs, which is *not* an SI base unit: 1 C is the total charge which passes a fixed point on a wire when a current of 1 A flows through that wire for 1 s. Hence, $1\,\mathrm{C} \equiv 1\,\mathrm{A\,s}$.

1.3 Newton's Laws

We shall treat these as axioms.

$\mathscr{N}$ ***I*** Every body continues in its state of rest or of uniform motion in a right [straight] line, unless it is compelled to change that state by a force impressed on it.

This is simply a special case of $\mathscr{N}$ ***II***.

$\mathscr{N}$ ***II*** The rate of change of motion [i.e., momentum: see below] is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If a particle has mass m and velocity $\mathbf{v}$ and is subjected to a total force $\mathbf{F}$ then its *momentum* is defined by $\mathbf{p} = m\mathbf{v}$, and $\mathscr{N}$ ***II*** states that

$$\boxed{\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}).}$$

If m is constant then $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a}$ is the acceleration.

$\mathscr{N}$ ***III*** The mutual actions of two bodies are always equal and directed to contrary parts.

That is, every action has an equal and opposite reaction; so if I press against a wall, applying a force $\mathbf{F}$ to it, then the wall will exert a force $-\mathbf{F}$ on me. Sometimes, however, it is less easy to work out how the reaction takes place: for instance, when a charged particle experiences a force because it is moving through a magnetic field. Nevertheless, the reaction still exists!

1.4 Reference Frames

Motion can only exist because we can detect it *relative* to something else. It would not make sense to say that an object is moving, or not moving, unless we know *what* it is moving, or not moving, relative to. However, this is cumbersome since most of the time it is obvious what we mean. It is therefore convenient to define a "standard" reference frame with respect to which we make all measurements (unless stated otherwise).

For our present purposes we shall consider the "fixed stars" (i.e., real stars, not the planets), which Newton assumed not to move, to be our standard reference frame. Any frame which moves at *constant velocity* relative to the fixed stars is called an *inertial frame*.

This idea is due to Ernst Mach (1836–1916) in 1883. Many refinements were made subsequently, leading eventually to the ideas of General Relativity in which the whole concept of a global inertial frame of this kind collapses.

Consider an inertial frame S in which the coordinate system is $(\mathbf{x}, t)$; and another inertial frame S' with coordinates $(\mathbf{x}', t')$. S' must be moving at a constant velocity, $\mathbf{V}$ say, relative to S; so

$$\mathbf{x}' = \mathbf{x} - \mathbf{V}t, \qquad t' = t. \tag{1.1}$$

Such a coordinate transformation is an example of a *Galilean transformation.* The full *Galilean group* of transformations also includes rotations of the spatial axes and changes in the origins of space and time; hence

$$x' = y - Vt, \quad y' = z, \quad z' = x - z_0, \quad t' = t - t_0$$

(where V, z_0 and t_0 are constants) is a Galilean transformation, because it consists of a rotation ($(x, y, z) \mapsto (y, z, x)$) followed by a constant velocity translation ($x \mapsto x - Vt$) and a fixed translation of origins ($z \mapsto z - z_0$, $t \mapsto t - t_0$).

We can check that the Galilean group is indeed a group, because we have inverses, an identity, etc.

Suppose that $\mathscr{N}\boldsymbol{II}$ is true in S, i.e., $\mathbf{F} = m\mathbf{a}$. With S' defined by (1.1),

$$
\begin{aligned}
\mathbf{v}' &= \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \mathbf{V} = \mathbf{v} - \mathbf{V} \\
\implies \qquad \mathbf{a}' &= \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t'} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{a}.
\end{aligned}
$$

Hence $\mathbf{F} = m\mathbf{a}'$, so $\mathscr{N}\boldsymbol{II}$ is true in S' as well. This is an example of the statement that the Laws of Physics must be equally true in any inertial frame.

Note that we can write (1.1) as

$$
\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -V_1 \\ 0 & 1 & 0 & -V_2 \\ 0 & 0 & 1 & -V_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \equiv \mathsf{M} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.
$$

We might ask what other transformation matrices M could be possible. They should certainly form a closed group. If we assume that the speed of light c is fixed (i.e., that $x = \pm ct \Rightarrow x' = \pm ct'$), then with a few very basic and obvious other assumptions — for instance, that if the speed of S' relative to S is positive, and a third frame S'' has positive speed relative to S', then the speed of S'' relative to S must also be positive — it can be proved that there is *only* one possibility! This is the Lorentz group of transformations of the form

$$
\mathbf{x}' = \gamma(\mathbf{x} - \mathbf{V}t), \qquad t' = \gamma\left(t - \frac{\mathbf{V}\cdot\mathbf{x}}{c^2}\right)
$$

where $\gamma = 1/\sqrt{1 - |\mathbf{V}|^2/c^2}$. This fact is the basis of Special Relativity.

In practical problems, we usually use the Earth (rather than the "fixed stars") as our reference frame; this ignores the rotation of the Earth (which produces an acceleration of the order of $2 \times 10^{-2}\,\mathrm{m/s^2}$, much less than gravitational acceleration). When we investigate the dynamics of the Solar System, we take the Sun as fixed, ignoring its rotation about the centre of the galaxy (an acceleration of order $2 \times 10^{-10}\,\mathrm{m/s^2}$).

1.5 Fundamental Forces

1.5.1 Gravity

Consider two particles, of masses m_1 and m_2 at $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be their relative displacement, and let $\hat{\mathbf{e}}_r = \mathbf{r}/|\mathbf{r}|$ be the unit vector in that direction. The first particle experiences a force of gravitational attraction towards the second given by Newton's Universal Law of Gravitation (so called because it acts between every pair of bodies in the Universe):

$$\boxed{\mathbf{F} = \frac{Gm_1m_2}{|\mathbf{r}|^2}\,\hat{\mathbf{e}}_r.}$$

The force on the second particle is equal and opposite. $\mathbf{F}$ can also be written in the form

$$\mathbf{F} = \frac{Gm_1m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1),$$

which is often more convenient for calculations even though it makes the inverse-square nature of the force less obvious. In these formulae, G is the *gravitational constant*, approximately $6.672 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$.

The ms which appear in this Law have a different physical significance from the m in $\mathscr{N}$ ***II***. These ms are *gravitational masses* and represent the quantity of gravitational matter. In $\mathscr{N}$ ***II***, m is the *inertial mass* and measures a body's resistance to being accelerated. It has been shown experimentally (first by Galileo, then using a torsion balance for the first time in 1889 by Baron Roland von Eötvös, and most recently to an accuracy of 10^{-18} in 2003) that gravitational mass is always proportional to inertial mass; hence, by choosing our units sensibly, we can arrange for them to be equal. So we need not distinguish.

Consider a fixed point R on the Earth's surface, with position vector $\mathbf{R}$ relative to the centre of the Earth; and a particle of mass m at a point $\mathbf{x}$ relative to R. In normal applications, $|\mathbf{x}| \ll |\mathbf{R}|$. Then if the mass of the Earth is M, the gravitational force on the particle is

$$\mathbf{F} = -\frac{GMm}{|\mathbf{R}+\mathbf{x}|^3}(\mathbf{R}+\mathbf{x}) \approx -\frac{GMm}{|\mathbf{R}|^3}\mathbf{R}.$$

We define $\mathbf{g} = -(GM/|\mathbf{R}|^3)\mathbf{R}$, so that $\mathbf{F} = m\mathbf{g}$. If the particle starts at $\mathbf{x}_0$ with velocity $\mathbf{v}_0$ and is subject to no other forces then

$$\begin{aligned} m\mathbf{a} = \mathbf{F} = m\mathbf{g} \qquad &\Longrightarrow \qquad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{g} \\ &\Longrightarrow \qquad \mathbf{v} = \mathbf{v}_0 + \mathbf{g}t \\ &\Longrightarrow \qquad \mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{g}t^2. \end{aligned}$$

We often write $\mathbf{g} = (0, 0, -g)^{\mathrm{T}}$ when the z-axis is measured vertically upwards from the Earth's surface.

Note that a body's *weight* is the force on it due to gravity, so is measured in Newtons, and is not the same as its *mass* measured in kilograms.

1.5.2 Electrostatics

Between any two particles with charges q_1 and q_2 at $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively there is an attractive or repulsive force. Using the same notations $\mathbf{r}$ and $\hat{\mathbf{e}}_r$ as in §1.5.1, the force on the first particle is

$$\boxed{-\frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}|^2}\,\hat{\mathbf{e}}_r} \qquad \text{or, equivalently,} \qquad -\frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

(which is towards the second particle iff q_1 and q_2 have opposite signs); here $1/4\pi\epsilon_0$ is a universal constant, approximately $8.99 \times 10^9\,\mathrm{N\,m^2/C^2}$. This *Coulomb force* is akin to gravity in many ways (it is inverse-square, and proportional to the product of the particles' charges) but instead may be repulsive.

Note that the electrostatic force is typically much stronger than gravity; for two electrons the ratio is around 4×10^{42}.

1.5.3 Electromagnetics

If a particle of charge q moves at velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ then it experiences a force (called the *Lorentz force*)

$$\boxed{q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).}$$

What is the origin of these fields? A fixed charge Q at the origin produces an electric field given by $\mathbf{E} = (Q/4\pi\epsilon_0|\mathbf{r}|^3)\mathbf{r}$, so the $q\mathbf{E}$ term is simply the Coulomb force. If Q moves, it creates an electric current, and currents produce magnetic fields.

1.6 Macroscopic Forces

1.6.1 Elasticity

Normal materials deform by $\lesssim 1\%$ before breaking (though, for example, rubber can deform by 400%); however, clever construction can amplify this greatly, as in a coiled spring.

Hooke's law (a linear approximation) applies to a spring extended a distance x:

$$\boxed{F = -kx}$$

is the force restoring the spring to its "natural length" l. Here k is the *spring constant* (with units $\mathrm{N\,m^{-1}} \equiv \mathrm{kg\,s^{-2}}$). Hooke described this relationship for "any springy body" and indeed it holds for elastic materials of many types, such as rubber bands and elastic strings.

k is related to the modulus of elasticity λ by $k = \lambda/l$.

The value of k depends on many variables, such as the temperature. Real materials have nonlinear force laws:

1.6.2 Friction

"Dry" Friction (a.k.a. "Coulomb friction")

This typically applies to one solid body in contact with another.
Let $\mathbf{N}$ be the normal reaction and $\mathbf{F}$ the external applied force;
then there will be a frictional force $\mathbf{R}$ opposing the motion that
would otherwise occur. In fact, $\mathbf{R} = -\mathbf{F}$ so long as $|\mathbf{F}| \leqslant \mu|\mathbf{N}|$ where μ is the coefficient of static friction (typically around 0.3). If this limit is exceeded, however, the body will start to move; when this occurs,

$$\boxed{|\mathbf{R}| = \mu'|\mathbf{N}|}$$

where μ' is the coefficient of dynamic friction (typically slightly less than μ). The frictional force will oppose the body's motion, so

$$\mathbf{R} = -\mu'|\mathbf{N}|\,\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Quadratic Friction

A good model for a solid body moving through still liquid or gas (e.g., air), rather than sliding against another solid body, is

$$|\mathbf{R}| \propto |\mathbf{v}|^2;$$

i.e.,

$$\boxed{\mathbf{R} = -k|\mathbf{v}|\mathbf{v}}$$

where k is a constant (depending on the density and viscosity of the fluid).

Linear Friction

At very low speeds a linear model may be more appropriate, i.e.,

$$\boxed{\mathbf{R} = -k'\mathbf{v}.}$$

1.6.3 Tension

In a string (or wire, rope, rod, etc.) placed under tension, every part of the string experiences tensional forces of magnitude T, say, pulling in opposite directions along the string. At the ends of the string, therefore, by $\mathscr{N}$ ***III*** there must also be forces of magnitude T pulling on whatever the string is attached to (a wall, particle, etc.). Normally we do not need to consider the internal forces in the string, but only the resulting forces on the end-points.

The tensional force need not in fact be of uniform magnitude along the length of the string: indeed, for a string of total mass m hanging vertically downwards, the tension at the top of the string must be greater by mg than the tension at the bottom of the string in order to support the string itself. We usually consider only *light* strings, i.e., mathematically ideal strings with zero mass; to move such a string requires zero force, so the tension must be constant along its length even if the string is moving.

A string passing over a pulley or similar arrangement will exert forces on each side of the pulley, with magnitudes T_1 and T_2 say. If the pulley is both light and smooth (so that no force is required to turn it, and no frictional forces are present) then the forces on each side must balance exactly, so $T_1 = T_2$.

Note that if the tension appears to become negative then the string in fact becomes slack and exerts no force. This is not true of a rod, which can exert "negative tension" known as *thrust*.

1.6.4 Constraints

A particle moving along a wire experiences a constraint force $\mathbf{N}$, known as the *normal reaction*, which is perpendicular to the wire. The only effect of this force is to ensure that the particle follows the constrained path. (There may well be other forces, e.g. friction $\mathbf{R}$, which are *not* perpendicular to the wire.)

The same applies to a particle moving on a surface (e.g., the outside of a sphere). Note that $\mathbf{N}$ always points *away* from the surface; if calculations ever show it pointing inwards then the particle will actually have flown off.

A particle attached by a rod to a fixed point O experiences a tension force $\mathbf{T}$, directed along the rod, constraining it to stay a fixed distance from O. The tension is therefore a constraint force.

In all three cases, note that the constraint force is always normal to the particle's velocity $\mathbf{v}$.

For two (or more) particles, the constraint forces are not necessarily perpendicular to their individual velocities. Suppose that two particles connected by a rigid rod of length l have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$. Then

$$|\mathbf{r}_2 - \mathbf{r}_1| = l \qquad \Longrightarrow \qquad (\mathbf{r}_2 - \mathbf{r}_1) \mathbin{\boldsymbol{.}} (\mathbf{r}_2 - \mathbf{r}_1) = l^2;$$

so, by differentiation with respect to time,

$$2(\mathbf{r}_2 - \mathbf{r}_1) \mathbin{\boldsymbol{.}} (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1) = 0,$$

i.e.,

$$(\mathbf{v}_1 - \mathbf{v}_2) \mathbin{\boldsymbol{.}} \mathbf{r} = 0$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the relative position vector. Since $\mathbf{T}$ is parallel to $\mathbf{r}$, we see that the constraint force is normal to the relative velocity, but not necessarily to either individual velocity.