Chapter 4

Energy and Stability

4.1 Energy in 1D

Consider a particle in 1D at position $x(t)$, subject to a force $F(x)$, so that $m\ddot{x} = F(x)$. Define the *kinetic energy* to be

$$\boxed{T = \tfrac{1}{2}m\dot{x}^2}$$

and the *potential energy* to be

$$\boxed{V(x) = -\int_a^x F(x')\,\mathrm{d}x'}$$

where a is an arbitrary constant. A different choice $\hat{a}$ would lead to a different potential energy

$$\begin{aligned}\hat{V}(x) &= -\int_{\hat{a}}^x F(x')\,\mathrm{d}x' \\ &= -\int_{\hat{a}}^a F(x')\,\mathrm{d}x' + V(x),\end{aligned}$$

in effect just adding a fixed constant on to V. We often choose $a = 0$ or ∞.

Note that $V'(x) = -F(x)$, and that this is true regardless of the choice of a.

The dimensions of energy are ML^2T^{-2} (either from $[T] = M(LT^{-1})^2$, or from $[V] = [\text{force}] \times L = (MLT^{-2})L$). In the SI system, it is measured in Joules ($1\,\mathrm{J} \equiv 1\,\mathrm{kg\,m^2/s^2}$).

Examples: a particle moving vertically under gravity has $m\ddot{z} = -mg$, so the potential energy is $V(z) = -\int_a^z(-mg)\,\mathrm{d}z' = mgz + \text{const.}$. A particle attached to a spring with spring constant k has $F = -kx$, so $V(x) = -\int_a^x(-kx')\,\mathrm{d}x' = \frac{1}{2}kx^2 + \text{const.}$

Multiply the equation of motion, $F = m\ddot{x}$, by $\dot{x}$:

$$\begin{aligned} m\dot{x}\ddot{x} &= \dot{x}F(x) \\ &= -\dot{x}V'(x) \\ &= -\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}V}{\mathrm{d}x} \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}m\dot{x}^2) &= -\frac{\mathrm{d}V}{\mathrm{d}t} \\ \implies \quad T + V &= \text{const.} \end{aligned}$$

Hence $\boxed{\text{the total energy } E = T + V \text{ is conserved.}}$

Note that if the forcing F depends on anything other than the position, x, then energy may not be conserved. For instance, in the damped simple harmonic motion example of §2.2, where a particle attached to a spring was subject to linear friction, we had

$$m\ddot{x} = -kx - c\dot{x}.$$

The potential energy resulting from the spring is $\frac{1}{2}kx^2$ (+ const.), as above, but it is not possible to define a potential energy for the damping term $c\dot{x}$. The total energy (kinetic and potential) combined is therefore $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$, but

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2) &= m\dot{x}\ddot{x} + kx\dot{x} \\ &= \dot{x}(m\ddot{x} + kx) \\ &= -c\dot{x}^2 \leqslant 0. \end{aligned}$$

The damping causes dissipation of energy.

When the potential energy is proportional to the mass of the particle, as is the case for instance in vertical motion under gravity where $V(z) = mgz + \text{const.}$, it is sometimes useful to consider the potential energy *per unit mass*, which would here be just $V_u(z) = gz + \text{const.}$ The force on a particle of mass m is then given by $F(z) = -mV_u'(z)$. This can be a useful definition because it allows us to specify the gravitational field without having to know in advance the mass of the particle on which gravity will be acting. However, a major drawback is that there is, unfortunately, no consistent naming convention for "potential energy per unit mass": some mathematicians call it the "potential field", others just the "potential", even though yet others use the word "potential" as a shorthand for "potential energy". The potential confusion is great.

Using the Energy Integral

From

$$\tfrac{1}{2}m\dot{x}^2 + V(x) = E,$$

the "energy integral", we deduce

$$\dot{x} = \pm\sqrt{\frac{2(E-V)}{m}}.$$

(We have to decide the sign on a case-by-case basis using physical considerations.) So

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \pm\sqrt{\frac{m}{2(E-V)}}$$

$$\implies \qquad t = \int \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \pm\int\sqrt{\frac{m}{2(E-V)}}\,\mathrm{d}x.$$

Example: what is the period of *finite* (i.e., non-infinitesimal) oscillations of a simple pendulum? Suppose that we release a pendulum bob from rest at $\theta = \theta_0$. Multiply (2.4),

$$ml\ddot{\theta} = -mg\sin\theta,$$

by $\dot{\theta}/m$ and integrate:

$$\begin{aligned} l\dot{\theta}\ddot{\theta} &= -g\dot{\theta}\sin\theta \\ \implies \qquad \tfrac{1}{2}l\dot{\theta}^2 &= g(\cos\theta - \cos\theta_0) \\ \implies \qquad \dot{\theta} &= \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}. \end{aligned}$$

To determine the period τ we consider a quarter-period in which θ increases from 0 to θ_0:

$$\int_0^{\frac{1}{4}\tau}\mathrm{d}t = \int_0^{\theta_0}\frac{\mathrm{d}\theta}{\dot{\theta}} \qquad \implies \qquad \tau = \sqrt{\frac{8l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

This is a "complete elliptic integral of the first kind".

In fact, $\tau = 4\sqrt{l/g}\,K(\sin\frac{1}{2}\theta_0)$ where K is the appropriate elliptic function.

4.2 Equilibria and Small Oscillations in 1D

A system has an *equilibrium* at $x = x_0$ if $F(x_0) = 0$, because then $x(t) = x_0\ \forall t$ is a solution of the equation of motion. Hence the equilibrium points are the *critical points* of V, i.e., where $V'(x_0) = 0$.

Consider a disturbance around an equilibrium point, $x = x_0 + \xi(t)$, where ξ is initially small. Then

$$\begin{aligned} m\ddot{x} &= F(x) = -V'(x_0 + \xi) \\ \implies \qquad m\ddot{\xi} &= -V'(x_0) - \xi V''(x_0) + O(\xi^2), \qquad \text{(Taylor series)} \end{aligned}$$

so that

$$\ddot{\xi} + \frac{V''(x_0)}{m}\xi = 0$$

to first order.

If $V''(x_0) < 0$ then the solution for ξ is

$$\xi = A\mathrm{e}^{\alpha t} + B\mathrm{e}^{-\alpha t}$$

where $\alpha = \sqrt{-V''(x_0)/m}$. Hence ξ grows larger as t increases (unless $A = 0$, which is only true for very special initial conditions), and so the solution does not necessarily stay in a small neighbourhood of x_0. Such a point is called an *unstable equilibrium*.

If $V''(x_0) > 0$ then instead ξ executes SHM with angular frequency

$$\boxed{\omega = \sqrt{\frac{V''(x_0)}{m}};}$$

hence ξ stays small and the solution stays in a small neighbourhood of x_0. Such a point is a *stable equilibrium*.

Hence a local maximum of V is an unstable point, while a minimum is stable. (A point at which $V''(x_0) = 0$ would require more detailed investigation, taking more terms in the Taylor series.)

Example: a particle on a spring. Here $V(x) = \frac{1}{2}kx^2$, so since $V'(0) = 0$ and $V''(0) = k > 0$, there is a stable equilibrium at $x = 0$. The frequency of oscillations is $\omega = \sqrt{k/m}$ and the period is $2\pi\sqrt{m/k}$, as previously found in §2.2.

4.3 Shape of the Potential Energy Function

Consider a graph of $V(x)$ as shown. We know that $T+V = E$, so $V = E - \frac{1}{2}m\dot{x}^2 \leqslant E$. Hence the particle is restricted to regions where $V(x) \leqslant E$.

Consider a particle at rest at the equilibrium point x_0 shown, where V has a local minimum; then $E = E_0 \equiv V(x_0)$. Suppose that the particle is now given a small disturbance. Such a disturbance will change the value of E, say to E_1 as shown. Then the particle is able to move, but is restricted to the region $[x_1, x_2]$; i.e., it must remain in a neighbourhood of x_0. This is an alternate way of showing that local minima of V are stable.

Now suppose that the particle is at another equilibrium point x_0' which is a local maximum of V. A small change in E, to E_1' say, does *not* restrict the motion of the particle to points near x_0'. So this is an unstable equilibrium.

Suppose that the particle is at x_0. What initial speed v (> 0) must we impart to it if it is to travel towards $+\infty$ and keep going? We need to ensure that $E > E_0' \equiv V(x_0')$, because otherwise the particle will not be able to reach x_0'; so

$$\tfrac{1}{2}mv^2 + V(x_0) > V(x_0') \qquad \Longrightarrow \qquad v > \sqrt{\frac{2\big(V(x_0') - V(x_0)\big)}{m}}.$$

Once it has reached x_0' it can keep going for ever.

4.4 Energy in 3D

Work

In 3D, the kinetic energy of a particle is given by

$$\boxed{T = \tfrac{1}{2}m|\dot{\mathbf{x}}|^2 = \tfrac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}.}$$

A force $\mathbf{F}$ acting on a particle which moves through $\delta\mathbf{x}$ is said to do *work* $\delta W = \mathbf{F} \cdot \delta\mathbf{x}$. The total work done by the force on a particle which moves from A to B is

$$\boxed{W = \int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{x}}$$

where the integral follows the path taken by the particle. It is obvious that in general W depends on the path taken.

The *power* is the rate of doing work, i.e., $P = \dot{W}$. In a time interval δt,

$$P = \frac{\delta W}{\delta t} = \frac{\mathbf{F} \cdot \delta\mathbf{x}}{\delta t} \qquad \Longrightarrow \qquad \boxed{P = \mathbf{F} \cdot \dot{\mathbf{x}}}$$

(in the limit $\delta t \to 0$). Note that from $\mathscr{N}\boldsymbol{II}$,

$$\begin{aligned} && \mathbf{F} &= m\ddot{\mathbf{x}} \\ \Longrightarrow && \mathbf{F} \cdot \dot{\mathbf{x}} &= m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \\ && &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\tfrac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}\big) \\ && &= \frac{\mathrm{d}T}{\mathrm{d}t}, \end{aligned}$$

so the power is the rate at which kinetic energy increases.

Forces which are normal to a particle's path do no work: this is because a particle's velocity $\dot{\mathbf{x}}$ is (by definition) tangential to its path, so $\mathbf{F} \,.\, \dot{\mathbf{x}} = 0$ and hence $P = 0$, i.e., the rate of doing work vanishes. For example, a magnetic field does no work on a charged particle, because $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$, and the field therefore neither increases nor decreases the particle's kinetic energy. Similarly, the tension in the string of a simple pendulum does no work on the bob.

Conservative Forces

A force field $\mathbf{F}(\mathbf{x})$ is said to be *conservative* iff the work done by the force on a particle moving from any point A to any other point B is independent of the path taken: i.e., iff $\int_A^B \mathbf{F} \,.\, \mathrm{d}\mathbf{x}$ is path-independent. We know from the *Vector Calculus* course that this is equivalent to saying that $\mathbf{F}$ is conservative iff

$$\boxed{\mathbf{F} = -\nabla V}$$

for some function $V(\mathbf{x})$, called the potential energy. (This is the 3D equivalent of $F = -V'(x)$ in 1D; in 1D *all* force fields $F(x)$ are conservative.)

But in 3D, not all force fields are conservative, because (as shown in the *Vector Calculus* course) the value of a line integral $\int_A^B \mathbf{F} \,.\, \mathrm{d}\mathbf{x}$ depends, in general, on the path taken.

What is the work done in moving a particle from a starting point A round a closed path back to the starting point? For a conservative force, the answer must be path-independent and must therefore be zero. But for a non-conservative force this does not apply and the force may have to do work just to get the particle back to where it started. This non-zero work done is generally dissipated, for example as heat.

For a conservative force, the work done is

$$W = \int_A^B \mathbf{F} \,.\, \mathrm{d}\mathbf{x} = -\int_A^B \nabla V \,.\, \mathrm{d}\mathbf{x} = -\int_A^B \mathrm{d}V = -\big[V(\mathbf{x})\big]_A^B,$$

i.e., equal to the decrease in potential energy. We can also prove that the total energy $E = T + V$ is conserved:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}m\dot{\mathbf{x}} \,.\, \dot{\mathbf{x}}) + \frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{x}) \\
&= m\dot{\mathbf{x}} \,.\, \ddot{\mathbf{x}} + \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= \dot{\mathbf{x}} \,.\, \mathbf{F} + \nabla V \,.\, \dot{\mathbf{x}} \\
&= 0.
\end{aligned}$$

Note that, just as in 1D, a local minimum of $V(\mathbf{x})$ is a stable equilibrium, and a local maximum is unstable — we can see this by considering the shape of $V(\mathbf{x})$ as in §4.3. (Quite easy to do in 2D; but almost impossible in 3D!) A saddle point of V is also unstable because the particle can move "downhill" from the saddle.

Examples of Potential Energy Functions in 3D

The force due to gravity on a particle of mass m near the Earth's surface is $\mathbf{F} = m\mathbf{g}$. But

$$\mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} = \nabla(-gz),$$

so $V(\mathbf{x}) = mgz + \text{const.}$: exactly the same answer as for 1D vertical motion.

A spring with spring constant k and natural length l attached to a fixed point O, but otherwise free to move in any direction in 3D, exerts a force

$$\mathbf{F}(\mathbf{r}) = -k(r-l)\hat{\mathbf{e}}_r$$

towards O, where $r = |\mathbf{r}|$ and $\hat{\mathbf{e}}_r = \mathbf{r}/r$ is the unit radial vector. We note that

$$\nabla\{\tfrac{1}{2}(r-l)^2\} = (r-l)\hat{\mathbf{e}}_r$$

so

$$V(\mathbf{r}) = \tfrac{1}{2}k(r-l)^2 + \text{const.},$$

the same as the 1D potential energy $\frac{1}{2}kx^2$ (+ const.) where x is the extension.

A uniform electric field $\mathbf{E}$ acting on a charge q produces a force $q\mathbf{E}$. But

$$\nabla(\mathbf{E}\,.\,\mathbf{x}) = \nabla(E_1x_1 + E_2x_2 + E_3x_3) = (E_1, E_2, E_3)^{\mathrm{T}} = \mathbf{E},$$

so the potential energy is $-q\mathbf{E}\,.\,\mathbf{x}$.

The gravitational force on a particle of mass m_2 with position vector $\mathbf{r}_2$ due to a particle of mass m_1 at $\mathbf{r}_1$ is

$$\mathbf{F} = -\frac{Gm_1m_2}{|\mathbf{r}|^2}\hat{\mathbf{e}}_r$$

from §1.5.1, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the relative position vector and $\hat{\mathbf{e}}_r = \mathbf{r}/|\mathbf{r}|$. So

$$V(\mathbf{r}) = -\frac{Gm_1m_2}{|\mathbf{r}|} \tag{4.1}$$

(where we choose the arbitrary constant so that $V = 0$ at infinity). In particular, the gravitational potential energy produced by the Earth (a mass M at the origin) acting on a particle of mass m at $\mathbf{r}$ is

$$V(\mathbf{r}) = -\frac{GMm}{r} \tag{4.2}$$

where $r = |\mathbf{r}|$.

In (4.1) we can consider the potential V to be a function of two variables $\mathbf{r}_1$, $\mathbf{r}_2$:

$$V(\mathbf{r}_1, \mathbf{r}_2) = -\frac{Gm_1m_2}{|\mathbf{r}_2 - \mathbf{r}_1|}.$$

The force on the second particle due to the first is then given by $-\nabla_2 V$ where ∇_2 denotes the gradient operator taken with respect to $\mathbf{r}_2$ (keeping $\mathbf{r}_1$ fixed), i.e.,

$$\nabla_2 \equiv \begin{pmatrix} \frac{\partial}{\partial(\mathbf{r}_2)_1} \\ \frac{\partial}{\partial(\mathbf{r}_2)_2} \\ \frac{\partial}{\partial(\mathbf{r}_2)_3} \end{pmatrix}.$$

The *same* potential function also gives us the force on the first particle due to the second, which is $-\nabla_1 V$; the symmetry in $\mathbf{r}_1$ and $\mathbf{r}_2$ ensures that the forces are equal and opposite. This idea can be extended to a system of n particles, with a potential function $V(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ that depends on all of the interparticle distances $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. The total force on the i^{th} particle due to the others is then given by $-\nabla_i V$.

4.5 Escape Velocity

The *escape velocity* is the minimum initial speed that would need to be imparted to a particle in a gravitational field to enable it to get arbitrarily far away.

For example, consider the potential energy (4.2) for a particle moving in the Earth's gravitational field, $-GMm/r$, as shown in the diagram. If the particle's total energy is $E_1 < 0$, then it is restricted to

$$r \leqslant r_1 = \frac{GMm}{-E_1};$$

if its total energy is instead $E_2 \geqslant 0$ then its motion is unrestricted and the particle can escape to ∞.

If the particle starts from $r = r_0$ with speed v then

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r_0}.$$

The escape velocity, i.e., the minimum value of v required to ensure that $E \geqslant 0$, is

$$v_{\text{escape}} = \sqrt{\frac{2GM}{r_0}}.$$

If a space ship starts on the surface of the Earth at $r = R$, then using $g = GM/R^2$ (from §1.5.1) we obtain an escape velocity of

$$\sqrt{2gR} \approx 11.2\,\text{km/s}$$

required to clear the Earth's gravitational field.

4.6 Motion with One Degree of Freedom

A particle may follow a trajectory in three dimensions which can be described by a single parameter $q(t)$. For example, the location of a bead moving along a wire can either be described by its position vector $\mathbf{r}$ in 3D, or instead by the scalar variable s defined as the arc-length measured along the wire from a fixed point. So long as we know the shape of the wire, the single variable s tells us everything we need to know about the bead's position. Such a system is known as having *one degree of freedom*, and we can treat it as being effectively one-dimensional.

For such a system, described by a parameter $q(t)$, we can apply all the methods of §§4.1–4.3 for motion in 1D, but we first need to obtain an equation of motion in the form

$$m\ddot{q} = F^*(q)$$

for some function $F^*(q)$ (which is not necessarily the actual force). Then, just as in §4.1, we can define a (pseudo-)kinetic energy $T^* = \frac{1}{2}m\dot{q}^2$ and a (pseudo-)potential energy $V^* = -\int_a^q F^*(q')\,\mathrm{d}q'$, and deduce that

$$T^* + V^* = E^*$$

is constant. Note however that T^* and V^* may well not be equal to the true kinetic and potential energies of the system.

For example, consider a simple pendulum swinging in a plane. This system has one degree of freedom, because the bob's position in 3D is specified completely by $\theta(t)$. We therefore have a choice of approaches:

- Treat the pendulum as a system with one degree of freedom. Starting from the equation of motion (2.4) in the appropriate form,

  $$m\ddot{\theta} = -\frac{mg}{l}\sin\theta \equiv F^*(\theta),$$

  define $T^* = \frac{1}{2}m\dot{\theta}^2$ and

  $$V^* = -\int_a^\theta F^*(\theta')\,\mathrm{d}\theta' = \frac{mg}{l}\int_a^\theta \sin\theta'\,\mathrm{d}\theta' = -\frac{mg}{l}\cos\theta + \text{const.}$$

  Ignoring the arbitrary constant, we therefore have that

  $$E^* = T^* + V^* = \tfrac{1}{2}m\dot{\theta}^2 - \frac{mg}{l}\cos\theta$$

  is constant. This approach is most useful when we are able to write down the equation of motion straight away but we do not know the true energy E of the system; it allows us to find an conserved quantity (namely E^*). Small oscillations can be investigated using the results of §4.2 $\big($including the formula for the frequency, $\omega = \sqrt{V^{*\prime\prime}(q_0)/m}\big)$ directly.

- Use the full 3D system. The speed of the bob is $l\dot{\theta}$, so the (true) kinetic energy of the system is $T = \frac{1}{2}ml^2\dot{\theta}^2$. The bob is at a height $l\cos\theta$ below the point of suspension O, so the (true) potential energy is $V = -mgl\cos\theta$. We therefore deduce that

$$E = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

is constant. To obtain the equation of motion, differentiate with respect to time:

$$0 = ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta \qquad \Longrightarrow \qquad \ddot{\theta} = -\frac{g}{l}\sin\theta.$$

This approach is most useful when we *cannot* intially write down the equation of motion of the system, but we *can* calculate its energy instead; the steps above then lead us to the equation of motion. To calculate the frequency of small oscillations about stable equilibria, it is necessary to consider a small disturbance and expand the equation of motion using Taylor Series as in §4.2: the formula given there for the frequency cannot be applied directly.

These two approaches are entirely consistent, because E and E^* differ only by a constant factor: $E^* = E/l^2$.

A general system with one degree of freedom has $\mathbf{x} = \mathbf{x}(q)$, so that

$$\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}t} = \mathbf{x}'\dot{q}$$

where a prime denotes differentiation with respect to q. The (true) kinetic energy is therefore

$$T = \tfrac{1}{2}m|\dot{\mathbf{x}}|^2 = \tfrac{1}{2}m|\mathbf{x}'|^2\dot{q}^2 = |\mathbf{x}'|^2T^*.$$

Thus the factor relating T to T^* is $|\mathbf{x}'|^2$.

In the case of a pendulum, it is obvious that $|\mathbf{x}'| = |\mathrm{d}\mathbf{x}/\mathrm{d}\theta| = l$, and the factor relating T to T^* is l^2 as found above.
