Chapter 5

Impulses and Collisions

5.1 Impulses

An *impulse* $\mathbf{I}$, applied at some time t_0, is defined to be a force $\mathbf{F}(t) = \mathbf{I}\,\delta(t - t_0)$ where δ is the Dirac delta-function. Physically, $\mathbf{I}$ is the mathematical idealisation of a very large force applied for a very short time. By $\mathscr{N}\boldsymbol{II}$,

$$\left[m\mathbf{v}\right]_{t_0^-}^{t_0^+} = \int_{t_0^-}^{t_0^+} \mathbf{F}\,\mathrm{d}t = \int_{t_0^-}^{t_0^+} \mathbf{I}\,\delta(t - t_0)\,\mathrm{d}t$$

$$\Longrightarrow \qquad \boxed{\left[m\mathbf{v}\right]_{t_0^-}^{t_0^+} = \mathbf{I},}$$

i.e., there is an instantaneous change $\mathbf{I}$ in the momentum.

Example: two balls of equal mass m are at rest and are joined by a rigid rod. An impulse $\mathbf{I}$ is applied to one of the balls making an angle $\frac{3}{4}\pi$ with the rod. In what direction does the ball move?

The applied impulse induces an instantaneous impulsive tension $\mathbf{T}$. The constraint imposed by the rod is that the components of the two velocities along the rod must be equal, from §1.6.4. Hence

$$v_1 \cos\theta = v_2.$$

Using Cartesian coordinates (x, y) and considering the impulses on each ball,

$$(T, 0) = \big[m\mathbf{v}_2\big]_{t_0^-}^{t_0^+} = (mv_2, 0),$$

$$(I\cos\tfrac{\pi}{4} - T, I\sin\tfrac{\pi}{4}) = \big[m\mathbf{v}_1\big]_{t_0^-}^{t_0^+} = (mv_1\cos\theta, mv_1\sin\theta).$$

It is easy to deduce that $\tan\theta = 2$, i.e., $\theta \approx 63°$.

5.2 Collisions

When two solid bodies of masses m_1, m_2 collide there is a plane of contact Π and a normal $\mathbf{n}$ at the point of contact. The bodies exert instantaneous forces on each other parallel to $\mathbf{n}$, and by $\mathscr{N}$ ***III*** these impulses $\pm\mathbf{I}$ are equal and opposite. If the bodies are smooth then there is no force perpendicular to $\mathbf{n}$ (i.e., in the plane Π).

(This is clearly idealised: real bodies deform; the forces they exert are not instantaneous; and they are rough, producing non-zero frictional forces perpendicular to $\mathbf{n}$.)

Let $\mathbf{u}_1$, $\mathbf{u}_2$ be the velocities before the collision and $\mathbf{v}_1$, $\mathbf{v}_2$ the velocities after. Then

$$m_1(\mathbf{v}_1 - \mathbf{u}_1) = \mathbf{I},$$

$$m_2(\mathbf{v}_2 - \mathbf{u}_2) = -\mathbf{I}$$

so

$$\boxed{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{u}_1 + m_2\mathbf{u}_2;}$$

that is, total momentum is conserved.

We also observe that the component of $m_1(\mathbf{v}_1 - \mathbf{u}_1)$ that is perpendicular to $\mathbf{n}$ must vanish (since $\mathbf{I}$ is parallel to $\mathbf{n}$). Hence $\mathbf{u}_1$ and $\mathbf{v}_1$ have equal components perpendicular to $\mathbf{n}$ — i.e., the component of velocity of m_1 that is perpendicular to $\mathbf{n}$ is conserved. Similarly for m_2. So collisions do not affect components of velocity that are normal to the direction of the collision.

This can be written succinctly as $\mathbf{v}_1 \times \mathbf{n} = \mathbf{u}_1 \times \mathbf{n}$, $\mathbf{v}_2 \times \mathbf{n} = \mathbf{u}_2 \times \mathbf{n}$.

However, this gives us no information about the components parallel to $\mathbf{n}$, that is, in the direction of the collision. For these, we use an empirical (and approximate) rule: *Newton's law of restitution*. This states that

$$\boxed{(\mathbf{v}_2 - \mathbf{v}_1)\,.\,\mathbf{n} = -e(\mathbf{u}_2 - \mathbf{u}_1)\,.\,\mathbf{n}}$$

where e is the *coefficient of restitution* ($0 \leqslant e \leqslant 1$). In one dimension,

$$v_2 - v_1 = e(u_1 - u_2),$$

or, put another way, the speed at which the bodies move away from each other is e times the speed at which they approach.

A collision with $e = 1$ is known as *elastic* (or *perfectly elastic*); one with $e < 1$ is called *inelastic*.

5.3 The Centre of Mass Frame

Let $M = m_1 + m_2$ be the total mass, and define the *centre of mass* to be

$$\mathbf{X} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{M}.$$

Then let

$$\mathbf{V} \equiv \dot{\mathbf{X}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{M};$$

we note that $\mathbf{V}$ is unchanged in a collision (because total momentum is conserved). Hence $\mathbf{V}$ is a constant (in the absence of any external forces).

Consider position vectors relative to the centre of mass, i.e., $\mathbf{x}' = \mathbf{x} - \mathbf{X}$. This is the *centre of mass frame*. We note that

$$\dot{\mathbf{X}} = \mathbf{V} \qquad \Longrightarrow \qquad \mathbf{X} = \mathbf{V}t + \mathbf{X}_0$$

where $\mathbf{X}_0$ is a constant, so $\mathbf{x}' = \mathbf{x} - \mathbf{X}_0 - \mathbf{V}t$: hence changing to the centre of mass frame is a Galilean transformation with speed $\mathbf{V}$ (see §1.4). In this frame,

$$\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{V} = \frac{m_2}{M}(\mathbf{v}_1 - \mathbf{v}_2), \tag{5.1}$$

$$\mathbf{v}_2' = \mathbf{v}_2 - \mathbf{V} = -\frac{m_1}{M}(\mathbf{v}_1 - \mathbf{v}_2), \tag{5.2}$$

so we see that the momenta of the particles are always equal and opposite in the centre of mass frame (both before and after a collision), i.e.,

$$m_1\mathbf{v}_1' = -m_2\mathbf{v}_2'. \tag{5.3}$$

5.4 Energy in Collisions

The total kinetic energy of the particles after a collision is

$$\begin{aligned}
T_{\text{after}} &= \tfrac{1}{2}m_1\mathbf{v}_1^2 + \tfrac{1}{2}m_2\mathbf{v}_2^2 \\
&= \tfrac{1}{2}m_1(\mathbf{V}+\mathbf{v}_1')^2 + \tfrac{1}{2}m_2(\mathbf{V}+\mathbf{v}_2')^2 \\
&= \tfrac{1}{2}(m_1+m_2)\mathbf{V}^2 + m_1\mathbf{V}\,.\,\mathbf{v}_1' + m_2\mathbf{V}\,.\,\mathbf{v}_2' + \tfrac{1}{2}m_1\mathbf{v}_1'^2 + \tfrac{1}{2}m_2\mathbf{v}_2'^2 \\
&= \tfrac{1}{2}M\mathbf{V}^2 + \mathbf{V}\,.\,(m_1\mathbf{v}_1' + m_2\mathbf{v}_2') + \tfrac{1}{2}\frac{m_1m_2^2}{M^2}(\mathbf{v}_1-\mathbf{v}_2)^2 + \tfrac{1}{2}\frac{m_1^2m_2}{M^2}(\mathbf{v}_1-\mathbf{v}_2)^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(from (5.1) and (5.2))} \\
&= \tfrac{1}{2}M\mathbf{V}^2 + \tfrac{1}{2}\frac{m_1m_2}{M^2}(m_2+m_1)(\mathbf{v}_1-\mathbf{v}_2)^2 \qquad\qquad \text{(from (5.3))} \\
&= \tfrac{1}{2}M\mathbf{V}^2 + \tfrac{1}{2}\frac{m_1m_2}{M}|\mathbf{v}_1-\mathbf{v}_2|^2.
\end{aligned}$$

This can be regarded as the kinetic energy of the centre of mass ($\tfrac{1}{2}M\mathbf{V}^2$) plus a term due to the relative velocity $\mathbf{v}_1 - \mathbf{v}_2$. Similarly, the total kinetic energy before the collision is

$$T_{\text{before}} = \tfrac{1}{2}M\mathbf{V}^2 + \tfrac{1}{2}\frac{m_1m_2}{M}|\mathbf{u}_1-\mathbf{u}_2|^2.$$

Note that for any vector $\mathbf{x}$,

$$|\mathbf{x}|^2 = |\mathbf{x}_\parallel|^2 + |\mathbf{x}_\perp|^2$$

where $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$ are the components parallel and perpendicular respectively to some vector $\mathbf{n}$, by Pythagoras' Theorem. Taking $\mathbf{n}$ to be the normal at the point of contact of the collision, we know from §5.2 that

$$(\mathbf{v}_1-\mathbf{v}_2)_\perp = (\mathbf{u}_1-\mathbf{u}_2)_\perp \qquad \text{but} \qquad (\mathbf{v}_1-\mathbf{v}_2)_\parallel = -e(\mathbf{u}_1-\mathbf{u}_2)_\parallel.$$

Hence

$$\begin{aligned}
T_{\text{before}} - T_{\text{after}} &= \tfrac{1}{2}\frac{m_1m_2}{M}\left(|\mathbf{u}_1-\mathbf{u}_2|^2 - |\mathbf{v}_1-\mathbf{v}_2|^2\right) \\
&= \tfrac{1}{2}\frac{m_1m_2}{M}\left(\left|(\mathbf{u}_1-\mathbf{u}_2)_\parallel\right|^2 + \left|(\mathbf{u}_1-\mathbf{u}_2)_\perp\right|^2 - \left|(\mathbf{v}_1-\mathbf{v}_2)_\parallel\right|^2 - \left|(\mathbf{v}_1-\mathbf{v}_2)_\perp\right|^2\right) \\
&= \tfrac{1}{2}\frac{m_1m_2}{M}\left|(\mathbf{u}_1-\mathbf{u}_2)_\parallel\right|^2(1-e^2) \\
&= \frac{m_1m_2}{2M}(1-e^2)\{(\mathbf{u}_1-\mathbf{u}_2)\,.\,\mathbf{n}\}^2.
\end{aligned}$$

So in an elastic collision ($e = 1$) there is no loss of energy; whereas in an inelastic collision, $T_{\text{after}} < T_{\text{before}}$. This is in fact the real definition of the term "elastic". The kinetic energy lost goes into vibration, sound, heat, temporary deformation of the bodies, etc.

Example: A billiard ball collides elastically with an identical stationary ball. The impact is slightly oblique so that the balls move off afterwards at an angle. Show that their velocities after impact are orthogonal.

Let the balls each have mass m; let $\mathbf{u}$ be the velocity of the first ball before the impact; and let $\mathbf{v}_1$, $\mathbf{v}_2$ be the velocities afterwards. By conservation of total momentum,

$$m\mathbf{u} = m\mathbf{v}_1 + m\mathbf{v}_2.$$

Squaring each side, we obtain

$$\mathbf{u}^2 = \mathbf{v}_1^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_2^2.$$

But because the collision is elastic, kinetic energy is conserved; therefore

$$\tfrac{1}{2}m\mathbf{u}^2 = \tfrac{1}{2}m\mathbf{v}_1^2 + \tfrac{1}{2}m\mathbf{v}_2^2.$$

Combining these results, we see that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0,$$

that is, $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal as required.