

Chapter 8

Orbits

8.1 Conics

Conic sections — first studied in the abstract by the Greeks — are the curves formed by the intersection of a plane with a cone. Ignoring degenerate cases (such as a point, or pairs of straight lines) there are only three cases. By geometrical means we can show that they each obey the following simple rule.

All conics are defined by two positive real parameters: a (which determines the size) and e (the *eccentricity*, which determines the shape). The two *foci* are at $(\pm ae, 0)$ and the *directrices* are the two lines $x = \pm a/e$. The conic is the set of points which obey the *focus-directrix property*: the distance from the focus is e times the distance from the corresponding directrix.

- **Case 1: $0 < e < 1$**

From the focus-directrix property,

$$\begin{aligned}\sqrt{(x - ae)^2 + y^2} &= e \left(\frac{a}{e} - x \right) \\ \implies \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} &= 1.\end{aligned}$$

This is an *ellipse* with semi-major axis a and semi-minor axis $a\sqrt{1 - e^2}$.

We can also include the special case of a circle of radius a , which occurs in the limit $e \rightarrow 0$.

- **Case 2: $e > 1$**

The same method gives

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1;$$

this is a *hyperbola*.

- **Case 3: $e = 1$**

In this case, the relevant directrix that corresponds to the focus at $x = a$ is the one at $x = -a$. Then

$$\begin{aligned} \sqrt{(x - a)^2 + y^2} &= x + a \\ \implies y^2 &= 4ax. \end{aligned}$$

This is a *parabola*.

A more general conic has the form

$$ax^2 + by^2 + 2cxy + 2dx + 2ey + f = 0$$

for some $a, b, c, d, e, f \in \mathbb{R}$. By rotating the axes and shifting the origin (using ideas from the Part IB course *Linear Algebra*) we can always transform such an equation to one of the above three cases, unless the conic is degenerate.

Polar Coordinates

In any of the above cases, place the origin of polar coordinates at one of the foci and if necessary rotate the axes through an angle π to place the corresponding directrix on the right. Let the distance from the focus to the directrix be l/e for some l ; then from the focus–directrix property,

$$\begin{aligned} r &= e \left(\frac{l}{e} - r \cos \theta \right) \\ \implies r &= \frac{l}{1 + e \cos \theta}. \end{aligned} \tag{8.1}$$

We see that l is the value of y immediately above the focus, since when $\theta = \frac{1}{2}\pi$, $r = l$; l is known as the *semi-latus rectum*. Note also that from the definition of l ,

$$l/e = |a/e - ae| \implies l = a|1 - e^2| \tag{8.2}$$

for $e \neq 1$; in the special case when $e = 1$, $l = 2a$ (from “Case 3” above).

Asymptotes

Ellipses are bounded, but hyperbolae and parabolae are unbounded. They have asymptotes at $\theta = \pm\theta_a$: we can find θ_a by considering the values of θ for which $r \rightarrow \infty$. From (8.1) this occurs when $1 + e \cos \theta_a = 0$, i.e., $\theta_a = \cos^{-1}(-1/e)$. (Note that θ_a exists because $e \geq 1$ for a parabola or hyperbola.)

8.2 The Gravitational Potential of a Sphere

Consider a uniform sphere of mass M , radius a and density ρ (so that $M = \frac{4}{3}\pi\rho a^3$), made up of many small elements of volume dV at position \mathbf{r} relative to the centre of the sphere. What is the total gravitational potential acting on a particle of mass m at a point \mathbf{R} outside the sphere?

Each element dV contributes a gravitational potential

$$-\frac{G(\rho dV)m}{|\mathbf{R} - \mathbf{r}|}$$

(from (4.1) in §4.4), since the mass of the element is ρdV and the distance from the element to the particle is $|\mathbf{R} - \mathbf{r}|$. Use spherical polar coordinates (r, θ, ϕ) where the $\theta = 0$ direction is along \mathbf{R} . Then

$$|\mathbf{R} - \mathbf{r}|^2 = (\mathbf{R} - \mathbf{r}) \cdot (\mathbf{R} - \mathbf{r}) = R^2 + r^2 - 2Rr \cos \theta$$

where $R = |\mathbf{R}|$. Hence the total potential at \mathbf{R} is

$$\begin{aligned} & -Gm\rho \iiint_{r \leq a} \frac{dV}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}} \\ &= -Gm\rho \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^a \frac{r^2 \sin \theta}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}} dr d\theta d\phi \quad (\text{using the Jacobian}) \\ &= -2\pi Gm\rho \int_0^a \left[\frac{r}{R} \sqrt{R^2 + r^2 - 2Rr \cos \theta} \right]_0^{\pi} dr \\ &= -\frac{2\pi Gm\rho}{R} \int_0^a r (\sqrt{R^2 + r^2 + 2Rr} - \sqrt{R^2 + r^2 - 2Rr}) dr \\ &= -\frac{2\pi Gm\rho}{R} \int_0^a 2r^2 dr = -\frac{4\pi Gm\rho a^3}{3R} = -\frac{GMm}{R}. \end{aligned}$$

This is *exactly* the same as if the entire mass of the sphere were concentrated in a particle of mass M at the origin! So, for the purposes of its gravitational field, we may treat the sphere as if it were a point mass.

(We can easily obtain the same result if ρ is a function $\rho(r)$ of the radius only, i.e., if the body is spherically symmetric.)

Note that we assumed that $R > a$. If $R < a$ (e.g., a particle down a mine shaft on the Earth) then the result is different because $\sqrt{R^2 + r^2 - 2Rr} = \sqrt{(R - r)^2} = |R - r|$ instead of just $R - r$. In this case, it turns out that only the part of the sphere which lies in $r < R$ contributes to the force (i.e., the force is $G(\frac{4}{3}\pi R^3 \rho)m/R^2$ radially inwards).

8.3 Kepler's Laws

During the late 16th century, Tycho Brahe made very careful observations and tables of the positions of the planets, all without a telescope. Johannes Kepler noted patterns in these tables and in the early 17th century announced his three laws.

K I The course of a planet in the heavens is not a circle, but an oval path, perfectly elliptical.

K II Through the centre of the world (and in the business at hand, it is the Sun for Copernicus and the Earth for the others), let the straight line be drawn intersecting the planet. This chord sweeps out equal areas in equal times.

K III The proportion between the periodic times of any two planets is precisely the sesquialterate proportion of their mean distances. [I.e., $T \propto R^{3/2}$ where T is the period of the orbit and R the mean radius.]

8.4 The Shape of the Orbit

We start by noting that the result of §8.2 means that we may treat the Sun and planets as if they were particles. Next, we note that the Sun is around 3×10^5 times more massive than the Earth, so the acceleration of the Sun caused by a planet is negligible compared to the acceleration of the planet caused by the Sun: we may regard the Sun as fixed.

We now note that the gravitational attraction of the Sun is a central force; hence from §6.2, the planet moves in a plane (perpendicular to the constant angular momentum vector). We use plane polar coordinates centred on the Sun, and know from §6.1 that

$$h = r^2 \dot{\theta} \tag{8.3}$$

is constant and

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \quad (8.4)$$

where M is the mass of the Sun.

To solve these equations, we *could* try to follow the method of §6.1 to obtain the energy equation: substituting (8.3) into (8.4) and multiplying by \dot{r} ,

$$\begin{aligned} \dot{r}\ddot{r} - \frac{h^2}{r^3}\dot{r} &= -\frac{GM}{r^2}\dot{r} \\ \implies \frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} &= \frac{GM}{r} + \text{const.} \\ \implies \frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - \frac{GMm}{r} &= \text{const.} \equiv E, \end{aligned} \quad (8.5)$$

so that

$$\dot{r}^2 = \frac{2E}{m} + \frac{2GM}{r} - \frac{h^2}{r^2} \quad \implies \quad t = \int \frac{dr}{\pm\sqrt{\left(\frac{2E}{m} + \frac{2GM}{r} - \frac{h^2}{r^2}\right)}}.$$

However, whilst this approach does give us t as a function of r , and therefore enables us to calculate for example the period, it tells us nothing about the details of the orbit because the equations cannot be inverted to give us r as a function of t .

Therefore, instead of trying to find $r(t)$ and $\theta(t)$, we just find the *shape* of the orbit, $r(\theta)$. We can do this with a trick: substitute

$$\boxed{u = r^{-1}.}$$

Note that

$$\dot{\theta} = \frac{h}{r^2} = hu^2,$$

so

$$\frac{d}{dt} = \frac{d\theta}{dt} \frac{d}{d\theta} = hu^2 \frac{d}{d\theta}.$$

Hence

$$\begin{aligned} \dot{r} &= hu^2 \frac{d}{d\theta}(u^{-1}) = -h \frac{du}{d\theta}, \\ \ddot{r} &= hu^2 \frac{d}{d\theta} \left(-h \frac{du}{d\theta} \right) = -h^2 u^2 \frac{d^2 u}{d\theta^2} \end{aligned}$$

whence in (8.4),

$$\begin{aligned} -h^2 u^2 \frac{d^2 u}{d\theta^2} - u^{-1} h^2 u^4 &= -GM u^2 \\ \implies \boxed{\frac{d^2 u}{d\theta^2} + u} &= \frac{GM}{h^2}. \end{aligned}$$

This differential equation has solution

$$u = \frac{GM}{h^2} + A \cos(\theta - \theta_0)$$

where A and θ_0 are constants. But θ_0 just corresponds to a rotation of the axes in polar coordinates; without loss of generality we can choose the $\theta = 0$ direction in such a way that $\theta_0 = 0$. Let

$$l = \frac{h^2}{GM}$$

and $e = Al$; then

$$r = u^{-1} = \frac{l}{1 + e \cos \theta}$$

which we recognise from (8.1) as the shape of a conic.

Since planets travel (by definition) in bounded orbits, these must be ellipses. We placed the Sun at the origin of polar coordinates, so it lies at one of the foci. An *apse* refers either to the point closest to the Sun or to the one furthest from it, known as *perihelion* and *aphelion* respectively. Other heavenly bodies (e.g., comets) might travel on parabolae or hyperbolae with the Sun at a focus.

Mercury has $e \approx 0.2$ and Pluto has $e \approx 0.24$. All the other planets have $e < 0.1$, so the difference between the minor and major axes of their orbits is less than $\frac{1}{2}\%$: almost circular.

Note: this method worked because the radial force is proportional to u^2 . However, it also works if the force is proportional to u^3 , or some linear combination of u^2 and u^3 .

8.5 Kepler's Laws Explained

We have just proved $\mathcal{K} \mathbf{I}$, with the Sun at one of the foci. Elliptical orbits imply an inverse-square law of gravitation, and conversely.

If a planet moves through a polar angle $\delta\theta$, then the area swept out by the radius vector (joining the Sun to the planet) is $\delta A = \frac{1}{2}r^2\delta\theta$ (by considering an infinitesimal triangle). Hence

$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{d\theta}{dt} = \frac{1}{2}r^2 \frac{h}{r^2} = \frac{1}{2}h.$$

So area is swept out at a constant rate, proving $\mathcal{K II}$. Therefore $\mathcal{K II}$ is just a statement that angular momentum is conserved; it implies that gravity is a central force.

What is the period of a planet's elliptical orbit? We know from §8.1 that the ellipse has semi-major axis a and semi-minor axis $b = a\sqrt{1 - e^2}$, and that $l = a(1 - e^2)$ from (8.2); so the area of the ellipse is

$$\pi ab = \pi a^2 \sqrt{1 - e^2} = \frac{\pi l^2}{(1 - e^2)^{3/2}}.$$

Since area is swept out at a constant rate $\frac{1}{2}h$, the period T of the orbit must satisfy

$$\begin{aligned} \frac{1}{2}hT &= \frac{\pi l^2}{(1 - e^2)^{3/2}} \quad \implies \quad T = \frac{2\pi l^2}{h(1 - e^2)^{3/2}} \\ &= \frac{2\pi}{\sqrt{GM}} \left(\frac{l}{1 - e^2} \right)^{3/2} \end{aligned}$$

since $l = h^2/(GM)$ from the last section. If we define

$$R = \frac{1}{2}(r_{\min} + r_{\max}) = \frac{1}{2} \left(\frac{l}{1 + e} + \frac{l}{1 - e} \right) = \frac{l}{1 - e^2}$$

to be the mean of the perihelion and aphelion distances, then

$$T = \frac{2\pi}{\sqrt{GM}} R^{3/2} \propto R^{3/2},$$

verifying $\mathcal{K III}$ (so long as we interpret "mean distance" as R).

8.6 Energy of an Orbit

From §8.4 we have

$$r = \frac{l}{1 + e \cos \theta}, \quad u = \frac{1 + e \cos \theta}{l}, \quad \dot{r} = -h \frac{du}{d\theta} = \frac{eh}{l} \sin \theta$$

where $l = h^2/(GM)$. So in the energy equation (8.5),

$$\begin{aligned} E &= \frac{1}{2}m \left(\dot{r}^2 + \frac{h^2}{r^2} \right) - \frac{GMm}{r} \\ &= \frac{mh^2}{2l^2} (e^2 \sin^2 \theta + (1 + e \cos \theta)^2) - \frac{GMm}{l} (1 + e \cos \theta) \\ &= \frac{G^2 M^2 m}{2h^2} (e^2 + 1 + 2e \cos \theta) - \frac{G^2 M^2 m}{h^2} (1 + e \cos \theta) \end{aligned}$$

which simplifies to

$$\boxed{E = \frac{G^2 M^2 m}{2h^2} (e^2 - 1).} \quad (8.6)$$

It is useful to define a pseudo-potential

$$V^*(r) = \frac{mh^2}{2r^2} - \frac{GMm}{r};$$

then

$$\frac{1}{2}m\dot{r}^2 + V^*(r) = E.$$

Hence the orbit is confined to radii for which $E \geq V^*(r)$.

If $E > 0$, say $E = E_1$, then $r \geq r_1$, but there is no upper limit on r : the orbit is unbounded. However, if $E < 0$, say $E = E_0$, then $r_{\min} \leq r \leq r_{\max}$ as marked on the diagram and the orbit is bounded, with perihelion and aphelion corresponding to r_{\min} and r_{\max} respectively.

These results are consistent with equation (8.6). If $E > 0$ then $e > 1$, so the orbit is a hyperbola which is indeed unbounded. Similarly, if $E < 0$ then $e < 1$ and the orbit is a (bounded) ellipse. The case $E = 0$, $e = 1$ is the dividing case of a parabola when the orbit has *just* enough energy to reach ∞ .

8.7 Interplanetary Travel [non-examinable]

Suppose we wish to travel from Earth to Mars. First, our rocket needs to escape the gravitational pull of the Earth; we must therefore launch it at a speed of at least 11.2 km/s (see §4.5). This places it in an orbit about the Sun, at the same radius r_E as the Earth, travelling at the same speed as the Earth relative to the Sun: on average, around 29.8 km/s.

We now execute a *Hohmann transfer*. We fire boosters to place the rocket in an elliptical orbit for which $r_{\min} = r_E$, $r_{\max} = r_M$ (the radius of Mars' orbit); note that we shall assume here for the purposes of simplicity that both the Earth's and Mars' orbits may be taken to be approximately circular. Now $r_{\min} + r_{\max} = 2l/(1 - e^2)$ from §8.5, so

$$\begin{aligned} E &= -\frac{G^2M^2m}{2h^2} \frac{2l}{r_E + r_M} && \text{(from §8.6)} \\ &= -\frac{GMm}{r_E + r_M} \end{aligned}$$

since $l = h^2/(GM)$. If V_E is the initial speed of the rocket then

$$\frac{1}{2}mV_E^2 - \frac{GMm}{r_E} = -\frac{GMm}{r_E + r_M}$$

which gives a value of $V_E \approx 32.7$ km/s; i.e., the boosters need to increase the rocket's speed by around 3 km/s. On arrival at Mars, we fire boosters to decelerate to the same speed as Mars, and then descend to the surface.

Of course, for this to work we need Mars to be at the right place when we arrive. Writing $R = \frac{1}{2}(r_E + r_M)$, it will take the rocket a time $\frac{1}{2}T = (\pi/\sqrt{GM})R^{3/2}$ (about 259 days) to travel to Mars (from §8.5), and

Mars has approximate angular velocity $\sqrt{GM} r_M^{-3/2}$ (since its period is $(2\pi/\sqrt{GM})r_M^{3/2}$); so we need Mars to be at an orbital angle

$$\begin{aligned}\beta &= \pi - \frac{\sqrt{GM}}{r_M^{3/2}} \frac{\pi}{\sqrt{GM}} \left(\frac{r_E + r_M}{2} \right)^{3/2} \\ &= \pi - \frac{\pi}{2\sqrt{2}} \left(\frac{r_E + r_M}{r_M} \right)^{3/2}\end{aligned}$$

(about 0.77 radians or 44°) ahead of the Earth when we leave.

For a *Grand Tour* — a fly-by of several planets — we can gain significant increases in speed by using a “slingshot” around a large planet such as Jupiter. Here, our rocket flies close to Jupiter and comes within its gravitational field. Suppose that in a frame of reference moving at \mathbf{v}_J (Jupiter’s velocity relative to the Sun), the rocket approaches at velocity \mathbf{u} and leaves at \mathbf{v} . By conservation of energy in this frame, $|\mathbf{u}| = |\mathbf{v}|$. But in the Sun’s frame of reference, the velocity of approach is $\mathbf{v}_J + \mathbf{u}$ and that of separation is $\mathbf{v}_J + \mathbf{v}$, and these may have very different magnitudes; by choosing the angle of our fly-by appropriately we can arrange that the rocket’s speed is increased greatly without the need for the use of further boosters.

8.8 Rutherford Scattering

A particle with charge q and mass m moves in the electric field produced by a charge Q which is fixed at the origin. The mutual gravitational attraction of the charges is negligible compared with the electrostatic Coulomb force. Initially, the particle is approaching from ∞ at speed V along a path which, in the absence of Q , would pass a distance b from the origin: b is the *impact parameter*. Hence $E = \frac{1}{2}mV^2$ and $h = Vb$ (because $h = |\mathbf{r} \times \dot{\mathbf{r}}|$); conservation of energy and of angular momentum tell us that the particle will recede with the same V and b .

If $qQ < 0$ then the particle is attracted to the origin with a force $(qQ/(4\pi\epsilon_0 r^2))\hat{\mathbf{e}}_r$. Comparing this with the gravitational force, $(-GMm/r^2)\hat{\mathbf{e}}_r$, we see that all our previous results apply if we replace GM by $-qQ/(4\pi\epsilon_0 m)$. In particular, it is possible to obtain

the standard result that the angle through which the particle is deflected is given by

$$\phi = 2 \tan^{-1} \left(\frac{-qQ}{4\pi\epsilon_0 mV^2 b} \right) > 0.$$

If $qQ > 0$ then the force is repulsive. The formula for the angle of deflection ϕ is unchanged, but now $\phi < 0$; so the particle is *scattered* through an angle

$$\Phi = -\phi = 2 \tan^{-1} \frac{qQ}{4\pi\epsilon_0 mV^2 b} = 2 \tan^{-1} \frac{qQ}{8\pi\epsilon_0 Eb}.$$

A careful analysis of the trajectory shows that in the case $qQ > 0$ we have $r = l/(-1 + e \cos \theta)$ instead of $l/(1 + e \cos \theta)$. This is still a hyperbola, but it is the “other sheet” — i.e., on the opposite side of the directrix from the focus. But the result for ϕ is unaltered.

Now consider a uniform beam of many identical incoming particles travelling towards the fixed charge Q . The number of particles with impact parameters in the range $(b, b + \delta b)$ is proportional to the area $\delta\Sigma = 2\pi b \delta b$; they are scattered through a range of angles $(\Phi, \Phi + \delta\Phi)$. Let

$$k = \frac{qQ}{8\pi\epsilon_0 E}$$

so that

$$b = k \cot \frac{1}{2}\Phi,$$

and define

$$\delta\Omega = 2\pi \sin \Phi \delta\Phi;$$

Rutherford's scattering cross-section is then defined to be

$$\begin{aligned}\sigma(\Phi) &= \left| \frac{d\Sigma}{d\Omega} \right| = -\frac{b}{\sin \Phi} \frac{db}{d\Phi} \quad (\text{because } \Phi \text{ decreases as } b \text{ increases, so } \frac{db}{d\Phi} < 0) \\ &= -\frac{k \cot \frac{1}{2}\Phi}{2 \sin \frac{1}{2}\Phi \cos \frac{1}{2}\Phi} \left(-\frac{1}{2}k \operatorname{cosec}^2 \frac{1}{2}\Phi \right) \\ &= \left(\frac{qQ}{16\pi\epsilon_0 E} \right)^2 \operatorname{cosec}^4 \frac{1}{2}\Phi.\end{aligned}$$

Ernest (later Lord) Rutherford used this method to explain the way that α -particles are scattered by atoms. It led him in 1911 to the first model of the atom as a small, heavy nucleus surrounded by orbital electrons.

The definition of $d\Omega$ is motivated by the fact that the solid angle of a cone of angle Φ is $\Omega = 2\pi(1 - \cos \Phi)$.
