Worked Example
A Forced Damped Harmonic Oscillator

We wish to find the forced response – i.e., the particular integral – to the damped harmonic oscillator given by

$$m\ddot{x} + c\dot{x} + kx = F\cos\Omega t.$$

Note that

$$F\cos\Omega t = \mathrm{Re}(F\mathrm{e}^{\mathrm{i}\Omega t}),$$

so try

$$x = \mathrm{Re}(A\mathrm{e}^{\mathrm{i}\Omega t})$$

where $A \in \mathbb{C}$. Then $\dot{x} = \mathrm{Re}(\mathrm{i}A\Omega\mathrm{e}^{\mathrm{i}\Omega t})$, etc., so

$$\mathrm{Re}\big(A(-m\Omega^2 + \mathrm{i}c\Omega + k)\mathrm{e}^{\mathrm{i}\Omega t}\big) = \mathrm{Re}(F\mathrm{e}^{\mathrm{i}\Omega t}),$$

which is only true for all t if

$$A = \frac{F}{-m\Omega^2 + \mathrm{i}c\Omega + k}$$

(check this by considering $t = 0$ and $t = \frac{1}{2}\pi/\Omega$).

Define the natural frequency $\omega = \sqrt{k/m}$ and $\gamma = c/(2m)$; then

$$A = \frac{F/m}{\omega^2 - \Omega^2 + 2\mathrm{i}\gamma\Omega}.$$

Now

$$\begin{aligned} x = \mathrm{Re}(A\mathrm{e}^{\mathrm{i}\Omega t}) &= \mathrm{Re}(|A|\mathrm{e}^{\mathrm{i}\Omega t + \mathrm{i}\arg A}) \\ &= |A|\cos(\Omega t + \arg A), \end{aligned}$$

so the amplitude of the forced oscillations is

$$|A| = \frac{F/m}{\sqrt{(\omega^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}}.$$

The phase of x is $\arg A$ ahead of the forcing $F\cos\Omega t$, so the *phase lag* is

$$-\arg A = \tan^{-1}\left(\frac{2\gamma\Omega}{\omega^2 - \Omega^2}\right).$$

For lightly damped systems (i.e., $\gamma \ll \omega$), there is a strong peak of the amplitude near $\Omega = \omega$, i.e., when the forcing frequency Ω coincides with the natural frequency ω. This is *resonance*.