Worked Example
A Particle Moving Under a Central Force

A particle of mass m is attached to one end of a massless spring with spring constant k and natural length l on a smooth horizontal table. The other end of the spring is fixed to a point O on the table. Initially, the spring is straight and at its natural length; the particle is then given a velocity of magnitude v at right angles to the spring. Starting from the equation of motion, obtain the energy equation. What choice of v ensures that in the subsequent motion the spring reaches a length of $2l$ but no more?

The spring is always straight and connected to O. It is clear that this is therefore a central force problem, and we use polar coordinates centred at O, so that the length of the spring is r. The force produced by the spring has magnitude $k(r-l)$ directed towards O. Using standard results for acceleration in polar coordinates, the equation of motion is

$$-k(r-l) = m(\ddot{r} - r\dot{\theta}^2), \qquad (*)$$

$$0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = \frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}).$$

Hence $r^2\dot{\theta}$ is a constant, h say; substituting $\dot{\theta} = h/r^2$ into $(*)$ gives

$$-k(r-l) = m\left(\ddot{r} - \frac{h^2}{r^3}\right).$$

To obtain the energy equation, we multiply by $\dot{r}$ and integrate with respect to time:

$$-k(r-l)\dot{r} = m\left(\dot{r}\ddot{r} - \frac{h^2\dot{r}}{r^3}\right)$$

$$\Longrightarrow \qquad -\tfrac{1}{2}k(r-l)^2 = m\left(\tfrac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2}\right) + \text{const.},$$

from which we obtain

$$\tfrac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) + \tfrac{1}{2}k(r-l)^2 = E$$

where E, the total energy, is constant.

We note that initially $r = l$, $\dot{r} = 0$ and $r\dot{\theta} = v$; hence $h = lv$ and $E = \frac{1}{2}mv^2$. The maximum value of r occurs when $\dot{r} = 0$; we want this to be at $r = 2l$. Hence from the energy equation,

$$\tfrac{1}{2}m\frac{(lv)^2}{(2l)^2} + \tfrac{1}{2}kl^2 = \tfrac{1}{2}mv^2,$$

i.e., $v = \sqrt{4kl^2/(3m)}$.