Worked Example
A Particle Moving Under a Central Force

A particle of mass \( m \) is attached to one end of a massless spring with spring constant \( k \) and natural length \( l \) on a smooth horizontal table. The other end of the spring is fixed to a point \( O \) on the table. Initially, the spring is straight and at its natural length; the particle is then given a velocity of magnitude \( v \) at right angles to the spring. Starting from the equation of motion, obtain the energy equation.

What choice of \( v \) ensures that in the subsequent motion the spring reaches a length of \( 2l \) but no more?

The spring is always straight and connected to \( O \). It is clear that this is therefore a central force problem, and we use polar coordinates centred at \( O \), so that the length of the spring is \( r \). The force produced by the spring has magnitude \( k(r - l) \) directed towards \( O \). Using standard results for acceleration in polar coordinates, the equation of motion is

\[
-k(r - l) = m(\ddot{r} - r\dot{\theta}^2),
\]

\[
0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = \frac{m}{r} \frac{d}{dt}(r^2\dot{\theta}).
\]

Hence \( r^2\dot{\theta} \) is a constant, \( h \) say; substituting \( \dot{\theta} = h/r^2 \) into (*) gives

\[
-k(r - l) = m\left(\ddot{r} - \frac{h^2}{r^3}\right).
\]

To obtain the energy equation, we multiply by \( \dot{r} \) and integrate with respect to time:

\[
-k(r - l)\dot{r} = m\left(\dddot{r} - \frac{h^2}{r^3}\dot{r}\right)
\]

\[
\implies -\frac{1}{2}k(r - l)^2 = m\left(\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2}\right) + \text{const.,}
\]

from which we obtain

\[
\frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) + \frac{1}{2}k(r - l)^2 = E
\]

where \( E \), the total energy, is constant.

We note that initially \( r = l, \dot{r} = 0 \) and \( r\dot{\theta} = v \); hence \( h = lv \) and \( E = \frac{1}{2}mv^2 \). The maximum value of \( r \) occurs when \( \dot{r} = 0 \); we want this to be at \( r = 2l \). Hence from the energy equation,

\[
\frac{1}{2}m\left(\frac{(lv)^2}{(2l)^2}\right) + \frac{1}{2}kl^2 = \frac{1}{2}mv^2,
\]

i.e., \( v = \sqrt{4kl^2/(3m)} \).