A hoop of wire in the shape of a circle of radius \( a \) is mounted vertically and rotates at constant angular speed \( \omega \) about a vertical axis through its centre. A bead moves smoothly on the wire. Find the equilibrium positions.

Let \( m \) be the mass of the bead, \( \mathbf{x} \) be its position vector relative to the centre of the hoop, \( \theta \) be the angle between \( \mathbf{x} \) and the downwards vertical, and \( \mathbf{e}_r, \mathbf{e}_\theta \) be unit radial and tangential vectors respectively. In the rotating frame \( S' \) of the wire, the bead simply moves in a circle of radius \( a \), and therefore using standard results,

\[
\mathbf{x} = a\mathbf{e}_r, \quad \left( \frac{d\mathbf{x}}{dt} \right)_{S'} = a\dot{\theta}\mathbf{e}_\theta, \quad \left( \frac{d^2\mathbf{x}}{dt^2} \right)_{S'} = -a\ddot{\theta}\mathbf{e}_r + a\dddot{\theta}\mathbf{e}_\theta.
\]

The forces on the bead are gravity and a normal reaction force \( \mathbf{N} \) from the wire.

Letting \( \mathbf{\omega} \) be the angular velocity vector (vertically upwards with magnitude \( \omega \)), the equation of motion in the rotating frame is

\[
m\mathbf{g} + \mathbf{N} = m\left\{ \left( \frac{d^2\mathbf{x}}{dt^2} \right)_{S'} + 2\mathbf{\omega} \times \left( \frac{d\mathbf{x}}{dt} \right)_{S'} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{x}) \right\} = m\{ -a\ddot{\theta}\mathbf{e}_r + a\dddot{\theta}\mathbf{e}_\theta + 2\mathbf{\omega} \times a\theta\mathbf{e}_\theta + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{a}_r) \}
\]

\[
= m\{ -a\ddot{\theta}\mathbf{e}_r + a\dddot{\theta}\mathbf{e}_\theta + 2a\theta\mathbf{\omega} \times \mathbf{e}_\theta + a(\mathbf{\omega} \cdot \mathbf{e}_r)\mathbf{\omega} - a(\mathbf{\omega} \cdot \mathbf{\omega})\mathbf{e}_r \}.
\]

We now wish to eliminate \( \mathbf{N} \): since \( \mathbf{N} \) must be normal to the wire, \( \mathbf{N} \cdot \mathbf{e}_\theta = 0 \). (Note that \( \mathbf{N} \) may have non-zero components both in the \( \mathbf{e}_r \)-direction and perpendicular to the plane of the hoop.) So, dotting the equation of motion with \( \mathbf{e}_\theta \) we obtain

\[
m\mathbf{g} \cdot \mathbf{e}_\theta = m\{ a\dddot{\phi} + a(\mathbf{\omega} \cdot \mathbf{e}_r)(\mathbf{\omega} \cdot \mathbf{e}_\theta) \}
\]

\[
\implies g \cos(\pi + \theta) = a\dddot{\phi} + a(\mathbf{\omega} \cos(\pi - \theta))(\mathbf{\omega} \cos(\pi - \theta))
\]

\[
\implies -g \sin \theta = a\ddot{\phi} - a\omega^2 \sin \phi \cos \phi
\]

\[
\implies \ddot{\theta} = \omega^2 \sin \theta \cos \theta - \frac{g}{a} \sin \theta.
\]

Equilibrium points occur when \( \dot{\theta} = \ddot{\theta} = 0 \), i.e., at \( \theta = 0, \pi \) or \( \pm \theta_0 \) where

\[
\theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 a}\right).
\]

These latter equilibrium points exist only when \( \omega \geq \sqrt{g/a} \). We could examine their stability by making a small perturbation about the equilibrium, for instance setting \( \theta = \theta_0 + \varepsilon \).