Worked Example
A Bead on a Rotating Wire Hoop

A hoop of wire in the shape of a circle of radius a is mounted vertically and rotates at constant angular speed ω about a vertical axis through its centre. A bead moves smoothly on the wire. Find the equilibrium positions.

Let m be the mass of the bead, $\mathbf{x}$ be its position vector relative to the centre of the hoop, θ be the angle between $\mathbf{x}$ and the downwards vertical, and $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ be unit radial and tangential vectors respectively. In the rotating frame S' of the wire, the bead simply moves in a circle of radius a, and therefore using standard results,

$$\mathbf{x} = a\hat{\mathbf{e}}_r, \qquad \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)_{S'} = a\dot{\theta}\hat{\mathbf{e}}_\theta, \qquad \left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{S'} = -a\dot{\theta}^2\hat{\mathbf{e}}_r + a\ddot{\theta}\hat{\mathbf{e}}_\theta.$$

The forces on the bead are gravity and a normal reaction force $\mathbf{N}$ from the wire.

Letting $\boldsymbol{\omega}$ be the angular velocity vector (vertically upwards with magnitude ω), the equation of motion in the rotating frame is

$$\begin{aligned} m\mathbf{g} + \mathbf{N} &= m\left\{\left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})\right\} \\ &= m\{-a\dot{\theta}^2\hat{\mathbf{e}}_r + a\ddot{\theta}\hat{\mathbf{e}}_\theta + 2\boldsymbol{\omega} \times a\dot{\theta}\hat{\mathbf{e}}_\theta + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times a\hat{\mathbf{e}}_r)\} \\ &= m\{-a\dot{\theta}^2\hat{\mathbf{e}}_r + a\ddot{\theta}\hat{\mathbf{e}}_\theta + 2a\dot{\theta}\boldsymbol{\omega} \times \hat{\mathbf{e}}_\theta + a(\boldsymbol{\omega} \cdot \hat{\mathbf{e}}_r)\boldsymbol{\omega} - a(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\hat{\mathbf{e}}_r\}. \end{aligned}$$

We now wish to eliminate $\mathbf{N}$: since $\mathbf{N}$ must be normal to the wire, $\mathbf{N} \cdot \hat{\mathbf{e}}_\theta = 0$. (Note that $\mathbf{N}$ may have non-zero components both in the $\hat{\mathbf{e}}_r$-direction *and* perpendicular to the plane of the hoop.) So, dotting the equation of motion with $\hat{\mathbf{e}}_\theta$ we obtain

$$\begin{aligned} & m\mathbf{g} \cdot \hat{\mathbf{e}}_\theta = m\{a\ddot{\theta} + a(\boldsymbol{\omega} \cdot \hat{\mathbf{e}}_r)(\boldsymbol{\omega} \cdot \hat{\mathbf{e}}_\theta)\} \\ \implies \quad & g\cos(\tfrac{\pi}{2} + \theta) = a\ddot{\theta} + a\big(\omega\cos(\pi - \theta)\big)\big(\omega\cos(\tfrac{\pi}{2} - \theta)\big) \\ \implies \quad & -g\sin\theta = a\ddot{\theta} - a\omega^2\sin\theta\cos\theta \\ \implies \quad & \ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta. \end{aligned}$$

Equilibrium points occur when $\dot{\theta} = \ddot{\theta} = 0$, i.e., at $\theta = 0$, π or $\pm\theta_0$ where

$$\theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 a}\right).$$

These latter equilibrium points exist only when $\omega \geqslant \sqrt{g/a}$. We could examine their stability by making a small perturbation about the equilibrium, for instance setting $\theta = \theta_0 + \varepsilon$.