Worked Example
Deflection of a Projectile due to the Earth's Rotation

A projectile is launched vertically upwards at speed V from a point at latitude λ on the surface of the Earth, which rotates with angular velocity $\boldsymbol{\omega}$. To first order in $\omega = |\boldsymbol{\omega}|$, how far away from the launch point does the projectile land, and in what direction?

Let the particle's position relative to the centre of the Earth be $\mathbf{x} = \mathbf{R} + \mathbf{x}^*$, where $\mathbf{R}$ is the position vector of the launch point. To measure $\mathbf{x}^*$, take local axes at the Earth's surface with x^* East, y^* North and z^* vertically upwards (a right-handed set).

The only force acting is gravity, so using the equation of motion,

$$\left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) = \mathbf{g}$$

where S' is the rotating frame of the Earth. We note that

$$\left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right)_{S'} = \mathbf{0}$$

since $\mathbf{R}$ is fixed in S', so

$$\left(\frac{\mathrm{d}^2\mathbf{x}^*}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times (\mathbf{R} + \mathbf{x}^*)\big) = \mathbf{g}.$$

We may neglect the centripetal term $\boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times (\mathbf{R} + \mathbf{x}^*)\big)$ since it is of second order in ω. Taking components in the x^*, y^*, z^* directions, and noting that

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ \omega\cos\lambda \\ \omega\sin\lambda \end{pmatrix}, \qquad \left(\frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t}\right)_{S'} = \begin{pmatrix} \dot{x}^* \\ \dot{y}^* \\ \dot{z}^* \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

with respect to these axes, we obtain

$$\begin{aligned} \ddot{x}^* &= 2\omega(\dot{y}^*\sin\lambda - \dot{z}^*\cos\lambda), \\ \ddot{y}^* &= -2\omega\dot{x}^*\sin\lambda, \\ \ddot{z}^* &= -g + 2\omega\dot{x}^*\cos\lambda, \end{aligned}$$

with $x^* = y^* = z^* = 0$, $\dot{x}^* = \dot{y}^* = 0$, $\dot{z}^* = V$ at $t = 0$.

From now on, we drop the stars (they have served their purpose).

We use the method of *iteration*: first we solve the equations with $\omega = 0$, then we substitute this result into the right hand sides of the above equations to find the first order solution. If we wished, we could repeat this process to find the second order solution, and so on indefinitely to any given order of accuracy. (Note that we would have to restore the centripetal term first if we wished to obtain the second order solution.)

When $\omega = 0$ the solution is just $x = y = 0$, $z = Vt - \frac{1}{2}gt^2$. So at first order we solve

$$\ddot{x} = -2\omega(V - gt)\cos\lambda,$$
$$\ddot{y} = 0,$$
$$\ddot{z} = -g;$$

the solution is

$$x = -\omega t^2(V - \tfrac{1}{3}gt)\cos\lambda,$$
$$y = 0,$$
$$z = Vt - \tfrac{1}{2}gt^2.$$

When the projectile returns to Earth, $z = 0$ so $t = 2V/g$ and

$$x = -\frac{4\omega V^3}{3g^2}\cos\lambda.$$

So the projectile lands $(4\omega V^3\cos\lambda)/(3g^2)$ due West of the launch point.