Worked Example
A Cylinder Rolling Down a Slope

A cylinder of mass M and radius a rolls down a rough inclined plane of slope α. What is its acceleration down the slope?

Let X be the distance travelled by the centre of mass of the cylinder down the slope, and θ be the angle through which the cylinder has turned. "Rolling" means "not slipping"; so

$$X = a\theta \qquad (1)$$

by considering the arc length of the edge of the cylinder.

The forces on the cylinder are its weight, $M\mathbf{g}$, which acts at the centre of mass; a normal reaction $\mathbf{N}$ which acts at the contact point between the cylinder and the slope; and a frictional force $\mathbf{R}$ also acting at the contact point. Considering the rate of change of total momentum resolved down the slope,

$$M\ddot{X} = Mg\sin\alpha - R \qquad (2)$$

where $R = |\mathbf{R}|$. We now take moments about the centre of mass (note that this is a "safe" case). Neither $M\mathbf{g}$ nor $\mathbf{N}$ have any moment (because the lines of action of those forces pass through the centre of mass), but $\mathbf{R}$ produces a couple of magnitude aR. Hence

$$I\ddot{\theta} = aR \qquad (3)$$

where $I = \frac{1}{2}Ma^2$ is the moment of inertia of the cylinder about its axis.

Combining equations (1), (2) and (3) we obtain

$$M\ddot{X} = Mg\sin\alpha - \frac{I}{a^2}\ddot{X},$$

i.e.,

$$\ddot{X} = \tfrac{2}{3}g\sin\alpha.$$

This is the required acceleration ($\frac{2}{3}$ of the acceleration which gravity alone would have produced).

Note that $|\mathbf{R}| \leqslant \mu|\mathbf{N}|$, where μ is the coefficient of static friction ("static" rather than "dynamic" because the rolling point of contact has zero velocity). But $|\mathbf{N}| = Mg\cos\alpha$ (resolving perpendicular to the slope), and $R = \frac{1}{2}M\ddot{X}$ from (1) and (3). Hence

$$\mu \geqslant \frac{\ddot{X}}{2g\cos\alpha} = \tfrac{1}{3}\tan\alpha.$$

If μ is less than this value then the frictional force will be insufficient to sustain rolling, and the cylinder will begin to slip.

[An alternative approach is to take moments about the rolling point of contact. That is not to say about a point moving down the slope – which would not be "safe" – but rather about a point fixed in the body on the circumference of the cylinder. At each time t we choose that point which will, instantaneously, be at the contact point (so we are considering different points in the body at different times t). Because the rolling point of contact of a body always has (instantaneously) zero velocity, this is "safe".

Now $\mathbf{N}$ and $\mathbf{R}$ pass through the point of contact, so have no moment about it; the only couple is therefore $Mga\sin\alpha$ produced by the gravitational force. The moment of inertia about an axis along the edge of the cylinder is $\frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2$, by the parallel axis theorem. So

$$\tfrac{3}{2}Ma^2\ddot{\theta} = Mga\sin\alpha,$$

and using (1) we obtain

$$\ddot{X} = \tfrac{2}{3}g\sin\alpha$$

as before.]