Worked Example
A Bead Sliding on a Rotating Rod

A bead of mass m is threaded onto a smooth rod of the same mass and length l which is pivoted at its midpoint so that it can rotate freely in a horizontal plane. Initially the bead is very close to the pivot and the system is at rest, when the rod is given an impulse which sets it rotating with initial angular speed Ω. With what speed does the bead fly off the end of the rod?

One method would be to consider the motion of the rod and the motion of the bead separately, taking into account the normal reaction between the rod and bead. Instead we treat the normal reaction as an internal force in a system consisting of the bead (a particle) and the rod (a rigid body) combined.

The moment of inertia of the rod about its midpoint is $I = \frac{1}{12}ml^2$. When the rod has turned through an angle θ and the bead is a distance r from the pivot, the total angular momentum about the pivot is

$$H = I\dot{\theta} + mr^2\dot{\theta}$$

and the energy is

$$E = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

Both quantities are conserved (friction is absent, and there is no external couple since any forces act at the pivot itself), so $H = I\Omega$ and $E = \frac{1}{2}I\Omega^2$ from the initial conditions ($\dot{\theta} = \Omega$, $\dot{r} = 0$, $r \approx 0$). Hence when $r = \frac{1}{2}l$,

$$\dot{\theta} = \frac{H}{I + mr^2} = \tfrac{1}{4}\Omega$$

and

$$\dot{r}^2 = \frac{2E}{m} - \frac{I}{m}\dot{\theta}^2 - r^2\dot{\theta}^2 = \tfrac{1}{16}l^2\Omega^2.$$

The final velocity of the bead is therefore composed of a radial component $\dot{r} = \frac{1}{4}l\Omega$ and a tangential component $r\dot{\theta} = \frac{1}{8}l\Omega$. The speed is $\sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = \frac{\sqrt{5}}{8}l\Omega$.