# Worked Example
# A Rod Sliding Inside a Wire Loop

A uniform rod of length $2a$ slides around the inside of a smooth circular wire loop mounted in a vertical plane, with its end-points attached to the loop. Denote by $X$ the centre of the rod, and by $O$ the centre of the loop. Show that

$$\dot{\theta}^2 = \frac{6gb}{a^2 + 3b^2}(\cos\theta - \cos\alpha) \qquad (1)$$

where $b$ is the length $OX$, $\theta$ is the angle between $OX$ and the vertical and $\alpha$ is the maximum value of $\theta$. Find also the frequency of small oscillations about the stable equilibrium.

The moment of inertia of the rod about its mid-point $X$ is $I = \frac{1}{12}M(2a)^2 = \frac{1}{3}Ma^2$, where $M$ is the mass of the rod. The kinetic energy of the rod is therefore

$$\tfrac{1}{2}M(b\dot{\theta})^2 + \tfrac{1}{2}(\tfrac{1}{3}Ma^2)\dot{\theta}^2,$$

because the speed of the centre of mass $X$ is $b\dot{\theta}$ and the kinetic energy relative to $X$ is $\frac{1}{2}I\dot{\theta}^2$. (Note that, relative to the centre of mass, the rod is simply rotating in a vertical plane with $\theta$ being the angle between the rod and the horizontal.) The potential energy is $-Mgb\cos\theta$ (taking $O$ as the zero of potential energy). Combining these results,

$$\tfrac{1}{2}(\tfrac{1}{3}a^2 + b^2)\dot{\theta}^2 - gb\cos\theta = E/M$$

where $E$, the total energy, is constant. Evaluating $E$ at $\theta = \alpha$ (where $\dot{\theta} = 0$ since it is a maximum of $\theta$) we obtain $E/M = -gb\cos\alpha$. The result follows.

Differentiating (1) with respect to time and dividing by $\dot{\theta}$ we obtain

$$2\ddot{\theta} = -\frac{6gb}{a^2 + 3b^2}\sin\theta.$$

Equilibrium points (where $\ddot{\theta} = 0$) are therefore, as expected, at $\theta = 0$ and $\pi$, which are obviously stable and unstable respectively. For small oscillations in which $\theta \ll 1$, we approximate $\sin\theta$ by $\theta$, so that

$$\ddot{\theta} + \frac{3gb}{a^2 + 3b^2}\theta = 0;$$

the (angular) frequency of oscillations is therefore

$$\sqrt{\frac{3gb}{a^2 + 3b^2}}.$$