Worked Example
A Particle Moving under Gravity with Quadratic Friction

A ball of mass m is thrown vertically upwards at speed V and experiences both gravity and quadratic air resistance. How high does it go and how long does it take to get there?

Measure z vertically upwards from the launch point. Then

$$m\ddot{z} = -mg - k\dot{z}^2$$

where k is a constant. Let $v = \dot{z}$ be the ball's speed at time t; then

$$\begin{aligned} m\dot{v} &= -mg - kv^2 \\ \implies \qquad m\int \frac{\mathrm{d}v}{mg + kv^2} &= -\int \mathrm{d}t. \end{aligned}$$

Using the substitution $v = \sqrt{mg/k}\,\tan\theta$ gives

$$\frac{1}{\alpha}\tan^{-1}\left(\frac{\alpha v}{g}\right) = c - t$$

where $\alpha \equiv \sqrt{gk/m}$ and c is a constant of integration. Hence

$$v = \frac{g}{\alpha}\tan\alpha(c-t),$$

and using the initial conditions, $c = \alpha^{-1}\tan^{-1}(\alpha V/g)$. So

$$\begin{aligned} z &= \frac{g}{\alpha}\int_0^t \tan\alpha(c-t)\,\mathrm{d}t \\ &= \frac{g}{\alpha^2}\big[\ln\cos\alpha(c-t)\big]_0^t \\ &= \frac{g}{\alpha^2}\ln\frac{\cos\alpha(c-t)}{\cos\alpha c}. \end{aligned}$$

The maximum height h occurs when $v = 0$, i.e., when

$$t = c = \sqrt{\frac{m}{gk}}\,\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,V\right),$$

and so

$$\begin{aligned} h &= \frac{g}{\alpha^2}\ln\sec\alpha c \\ &= \frac{g}{\alpha^2}\ln\sqrt{1+\tan^2\alpha c} \\ &= \frac{m}{2k}\ln\left(1+\frac{kV^2}{mg}\right). \end{aligned}$$

Alternative Method

If the time taken is not required then we can use the chain rule to write

$$\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}z} = v\frac{\mathrm{d}v}{\mathrm{d}z}.$$

Then

$$\begin{aligned}
 mv\frac{\mathrm{d}v}{\mathrm{d}z} &= -mg - kv^2 \\
\implies \qquad \int_0^h \mathrm{d}z &= -\int_V^0 \frac{mv\,\mathrm{d}v}{mg + kv^2} \\
\implies \qquad h &= -\left[\frac{m}{2k}\ln(mg + kv^2)\right]_V^0 \\
&= \frac{m}{2k}\ln(mg + kV^2) - \frac{m}{2k}\ln(mg) \\
&= \frac{m}{2k}\ln\left(1 + \frac{kV^2}{mg}\right),
\end{aligned}$$

as before.

Descent

On the way down, the equation of motion is different because the retardation force is in the opposite direction:

$$m\ddot{z} = -mg + k\dot{z}^2$$

(where z still measures distance vertically *upwards*), so

$$m\int \frac{\mathrm{d}v}{-mg + kv^2} = \int \mathrm{d}t.$$

We use the substitution $v = \sqrt{mg/k}\tanh\theta$ to obtain

$$v = -\frac{g}{\alpha}\tanh\alpha(t - c)$$

(using the initial condition $v = 0$ at $t = c$) and

$$z = h - \frac{g}{\alpha}\ln\cosh\alpha(t - c).$$