Worked Example
A Struck Pendulum

A simple pendulum consists of a string of length l attached to a bob. It is hanging at rest when the bob is struck horizontally, giving it a speed V. Show that the pendulum will oscillate back and forth if $V^2/(gl) \leqslant 2$ or will rotate in a single direction if $V^2/(gl) \geqslant 5$. Show that for intermediate values, the string will go slack, and describe the motion.

Denote the angle of the bob to the vertical by θ. Resolving perpendicular to the string,

$$ml\ddot{\theta} = -mg\sin\theta$$

where m is the mass of the bob (because the tangential component of the acceleration of the bob is $2\dot{r}\dot{\theta} + r\ddot{\theta}$, but $r = l$ so $\dot{r} = 0$). Multiplying by $\dot{\theta}/m$ and integrating with respect to time,

$$l\dot{\theta}\ddot{\theta} = -g\dot{\theta}\sin\theta$$

$$\implies \qquad \tfrac{1}{2}l\dot{\theta}^2 = g\cos\theta + c.$$

The initial conditions are $\theta = 0$, $\dot{\theta} = V/l$, so $c = V^2/(2l) - g$.

Now let T be the tension in the string. Resolving parallel to the string and using the fact that the radial component of acceleration is $\ddot{r} - r\dot{\theta}^2$,

$$-ml\dot{\theta}^2 = mg\cos\theta - T \qquad \implies \qquad T = 3mg\cos\theta + 2mc$$

$$= mg(3\cos\theta - 2) + mV^2/l.$$

Hence $T = 0$ when $\theta = \theta_1$, where $\cos\theta_1 = \frac{2}{3} - V^2/(3gl)$; whereas $\dot{\theta} = 0$ when $\theta = \theta_0$, where $\cos\theta_0 = 1 - V^2/(2gl) = \frac{3}{2}\cos\theta_1$.

The behaviour of the pendulum depends on the values of θ_0 (which is the maximum value of θ) and θ_1 (the value of θ beyond which the string would go slack). There are three cases:

(i) $V^2/(gl) \leqslant 2$. Then $0 \leqslant \cos\theta_1 \leqslant \cos\theta_0 \leqslant 1$, i.e., $0 \leqslant \theta_0 \leqslant \theta_1 \leqslant \frac{1}{2}\pi$; hence the pendulum oscillates back and forth between $\pm\theta_0$ but never goes beyond θ_1, so the string never goes slack.

(ii) $V^2/(gl) > 5$. Then $\cos\theta_0 < \cos\theta_1 < -1$, so there are no real solutions for either θ_0 or θ_1. The bob goes "over the top" and the string never goes slack, so the pendulum continues to rotate in the same direction. If $V^2/(gl) = 5$ then $\theta_1 = \pi$, so the string momentarily has zero tension as the pendulum goes over the top; but the qualitative behaviour is the same.

(iii) $2 < V^2/(gl) < 5$. In this intermediate case, $-1 < \cos\theta_1 < 0$, i.e., $\frac{1}{2}\pi < \theta_1 < \pi$; and $\cos\theta_0 < \cos\theta_1 < 0$ so $\theta_0 > \theta_1$ (unless $\cos\theta_0 < -1$ in which case θ_0 does not even exist). Hence the bob reaches θ_1 before its theoretical maximum amplitude. But at θ_1 the string goes slack and our governing equations are no longer valid; the bob falls like a free projectile.