A simple pendulum consists of a string of length $l$ attached to a bob. It is hanging at rest when the bob is struck horizontally, giving it a speed $V$. Show that the pendulum will oscillate back and forth if $V^2/(gl) \leq 2$ or will rotate in a single direction if $V^2/(gl) \geq 5$. Show that for intermediate values, the string will go slack, and describe the motion.

Denote the angle of the bob to the vertical by $\theta$. Resolving perpendicular to the string, 

$$ml\ddot{\theta} = -mg \sin \theta$$

where $m$ is the mass of the bob (because the tangential component of the acceleration of the bob is $2\dot{r}\dot{\theta} + r\ddot{\theta}$, but $r = l$ so $\dot{r} = 0$). Multiplying by $\dot{\theta}/m$ and integrating with respect to time,

$$l\dot{\theta} \ddot{\theta} = -g \dot{\theta} \sin \theta \implies \frac{1}{2}l\dot{\theta}^2 = g \cos \theta + c.$$

The initial conditions are $\theta = 0$, $\dot{\theta} = V/l$, so $c = V^2/(2l) - g$.

Now let $T$ be the tension in the string. Resolving parallel to the string and using the fact that the radial component of acceleration is $\ddot{r} - r\dot{\theta}^2$,

$$-ml\dot{\theta}^2 = mg \cos \theta - T \implies T = 3mg \cos \theta + 2mc = mg(3 \cos \theta - 2) + mV^2/l.$$

Hence $T = 0$ when $\theta = \theta_1$, where $\cos \theta_1 = \frac{2}{3} - V^2/(3gl)$; whereas $\dot{\theta} = 0$ when $\theta = \theta_0$, where $\cos \theta_0 = 1 - V^2/(2gl) = \frac{3}{2} \cos \theta_1$.

The behaviour of the pendulum depends on the values of $\theta_0$ (which is the maximum value of $\theta$) and $\theta_1$ (the value of $\theta$ beyond which the string would go slack). There are three cases:

(i) $V^2/(gl) \leq 2$. Then $0 \leq \cos \theta_1 \leq \cos \theta_0 \leq 1$, i.e., $0 \leq \theta_0 \leq \theta_1 \leq \frac{1}{2} \pi$; hence the pendulum oscillates back and forth between $\pm \theta_0$ but never goes beyond $\theta_1$, so the string never goes slack.

(ii) $V^2/(gl) > 5$. Then $\cos \theta_0 < \cos \theta_1 < -1$, so there are no real solutions for either $\theta_0$ or $\theta_1$. The bob goes “over the top” and the string never goes slack, so the pendulum continues to rotate in the same direction. If $V^2/(gl) = 5$ then $\theta_1 = \pi$, so the string momentarily has zero tension as the pendulum goes over the top; but the qualitative behaviour is the same.

(iii) $2 < V^2/(gl) < 5$. In this intermediate case, $-1 < \cos \theta_1 < 0$, i.e., $\frac{1}{2} \pi < \theta_1 < \pi$; and $\cos \theta_0 < \cos \theta_1 < 0$ so $\theta_0 > \theta_1$ (unless $\cos \theta_0 < -1$ in which case $\theta_0$ does not even exist). Hence the bob reaches $\theta_1$ before its theoretical maximum amplitude. But at $\theta_1$ the string goes slack and our governing equations are no longer valid; the bob falls like a free projectile.