Worked Example
Small Oscillations with Elasticity

A particle of mass m is attached to four identical, light elastic strings with modulus of elasticity λ and natural length l. The other ends of the strings are attached to the four corners of a smooth square table-top of side $2l$. The particle is at rest at the centre of the table when it is displaced through a small distance in a direction parallel to one of the sides, whereupon it is released. What is the frequency of the subsequent oscillations?

Let the displacement of the particle at time t be y. Then the lengths of the two upper strings are d_1 where

$$d_1^2 = l^2 + (l-y)^2 = 2l^2 - 2ly + y^2$$

while the two lower strings have length d_2 where

$$d_2^2 = l^2 + (l+y)^2 = 2l^2 + 2ly + y^2.$$

The potential energy is therefore

$$\begin{aligned} V(y) &= \frac{\lambda}{2l}\{2(d_1-l)^2 + 2(d_2-l)^2\} \\ &= \frac{\lambda}{l}(d_1^2 + d_2^2 - 2ld_1 - 2ld_2 + 2l^2) \\ &= \frac{\lambda}{l}\big(6l^2 + 2y^2 - 2l\sqrt{2l^2 - 2ly + y^2} - 2l\sqrt{2l^2 + 2ly + y^2}\big) \\ &= \frac{2\lambda}{l}\big(y^2 - l\sqrt{2l^2 - 2ly + y^2} - l\sqrt{2l^2 + 2ly + y^2}\big) + \text{const.} \end{aligned}$$

Hence

$$V'(y) = \frac{2\lambda}{l}\left(2y - l\frac{y-l}{\sqrt{2l^2 - 2ly + y^2}} - l\frac{y+l}{\sqrt{2l^2 + 2ly + y^2}}\right),$$

so we observe that $V'(0) = 0$; that is, the centre of the square is indeed an equilibrium point.

We now calculate

$$V''(y) = \frac{2\lambda}{l}\left(2 - l\frac{l^2}{(2l^2 - 2ly + y^2)^{3/2}} - l\frac{l^2}{(2l^2 + 2ly + y^2)^{3/2}}\right).$$

Recall that the frequency of small oscillations is $\sqrt{V''(0)/m}$; in this case

$$\begin{aligned} V''(0) &= \frac{2\lambda}{l}\left(2 - \frac{l^3}{2\sqrt{2}\,l^3} - \frac{l^3}{2\sqrt{2}\,l^3}\right) \\ &= \frac{2\lambda}{l}\left(2 - \frac{1}{\sqrt{2}}\right) \\ &= \big(4 - \sqrt{2}\big)\lambda/l. \end{aligned}$$

Hence the frequency of the oscillations is $\sqrt{\big(4-\sqrt{2}\big)\lambda/(ml)}$.