Worked Example
Small Oscillations with Elasticity

A particle of mass $m$ is attached to four identical, light elastic strings with modulus of elasticity $\lambda$ and natural length $l$. The other ends of the strings are attached to the four corners of a smooth square table-top of side $2l$. The particle is at rest at the centre of the table when it is displaced through a small distance in a direction parallel to one of the sides, whereupon it is released. What is the frequency of the subsequent oscillations?

Let the displacement of the particle at time $t$ be $y$. Then the lengths of the two upper strings are $d_1$ where
$$d_1^2 = l^2 + (l - y)^2 = 2l^2 - 2ly + y^2$$
while the two lower strings have length $d_2$ where
$$d_2^2 = l^2 + (l + y)^2 = 2l^2 + 2ly + y^2.$$

The potential energy is therefore
$$V(y) = \frac{\lambda}{2l} \{2(d_1 - l)^2 + 2(d_2 - l)^2\}$$
$$= \frac{\lambda}{l} (d_1^2 + d_2^2 - 2ld_1 - 2ld_2 + 2l^2)$$
$$= \frac{\lambda}{l} (6l^2 + 2y^2 - 2l\sqrt{2l^2 - 2ly + y^2} - 2l\sqrt{2l^2 + 2ly + y^2})$$
$$= \frac{2\lambda}{l} (y^2 - l\sqrt{2l^2 - 2ly + y^2} - l\sqrt{2l^2 + 2ly + y^2}) + \text{const.}$$

Hence
$$V'(y) = \frac{2\lambda}{l} \left(2y - l\frac{y - l}{\sqrt{2l^2 - 2ly + y^2}} - l\frac{y + l}{\sqrt{2l^2 + 2ly + y^2}}\right),$$
so we observe that $V'(0) = 0$; that is, the centre of the square is indeed an equilibrium point.

We now calculate
$$V''(y) = \frac{2\lambda}{l} \left(2 - l\frac{l^2}{(2l^2 - 2ly + y^2)^{3/2}} - l\frac{l^2}{(2l^2 + 2ly + y^2)^{3/2}}\right).$$
Recall that the frequency of small oscillations is $\sqrt{V''(0)/m}$; in this case
$$V''(0) = \frac{2\lambda}{l} \left(2 - \frac{l^3}{2\sqrt{2}l^3} - \frac{l^3}{2\sqrt{2}l^3}\right)$$
$$= \frac{2\lambda}{l} \left(2 - \frac{1}{\sqrt{2}}\right)$$
$$= (4 - \sqrt{2})\lambda/l.$$

Hence the frequency of the oscillations is $\sqrt{(4 - \sqrt{2})\lambda/(ml)}$. 