**Worked Example**

**The Speed of a Charged Particle in an Electromagnetic Field**

A particle of mass $m$ and charge $q$ moves in a constant uniform electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. Gravity may be ignored. Write down an expression for the particle’s energy and show that it is conserved. Prove that if the particle starts from $x_0$ with speed $u$, and a time $t$ later is at $x$ with speed $v$, then

$$v^2 = u^2 + 2a \cdot s$$

where $s = x - x_0$ for a suitable constant vector $a$. Show further that

$$v \leq u + (q/m)|\mathbf{E}|t.$$

**Method 1:** The potential energy corresponding to the electric field is $-q\mathbf{E} \cdot x$. However, there is no potential energy corresponding to the magnetic field, since the magnetic field does no work on the particle. Therefore the energy $E$ is given by

$$E = \frac{1}{2}m|\dot{x}|^2 - q\mathbf{E} \cdot x.$$

To show that it is conserved, differentiate with respect to time:

$$\dot{E} = \frac{1}{2}m \frac{d}{dt} (\dot{x} \cdot \dot{x}) - q\mathbf{E} \cdot \dot{x}$$

$$= m\ddot{x} \cdot \dot{x} - q\mathbf{E} \cdot \dot{x}.$$

But the equation of motion for the particle is given by the Lorentz force,

$$m\ddot{x} = q(\mathbf{E} + \dot{x} \times \mathbf{B}),$$

so

$$\dot{E} = q\dot{x} \cdot (\mathbf{E} + \dot{x} \times \mathbf{B}) - q\mathbf{E} \cdot \dot{x} = 0,$$

as required.

**Method 2:** Starting from the equation of motion (*), dot with $\dot{x}$ and integrate:

$$m\ddot{x} = q(\mathbf{E} + \dot{x} \times \mathbf{B})$$

$$\implies m\dot{x} \cdot \ddot{x} = q\dot{x} \cdot \mathbf{E}$$

$$\implies \frac{1}{2}m\dot{x} \cdot \ddot{x} = q\dot{x} \cdot \mathbf{E} + \text{const.},$$

so

$$\frac{1}{2}m|\dot{x}|^2 - q\mathbf{E} \cdot x = E$$

is constant, as before.

Now, having used either method for the first part, use conservation of $E$ between times 0 and $t$:

$$\frac{1}{2}mv^2 - q\mathbf{E} \cdot x = \frac{1}{2}mu^2 - q\mathbf{E} \cdot x_0$$

$$\implies v^2 = u^2 + \frac{2q}{m} \mathbf{E} \cdot (x - x_0),$$

which is of the required form with $a = (q/m)\mathbf{E}$. 
For the last part, write $E$ in terms of $v = |\dot{x}|$ as follows and differentiate:

\[ E = \frac{1}{2}mv^2 - qE \cdot x \]

\[ \implies 0 = mv\dot{v} - qE \cdot \dot{x} \]

\[ \implies mv\dot{v} = qE \cdot \dot{x} \]

\[ \leq q|E||\dot{x}| \]

\[ = q|E|v \]

\[ \implies \dot{v} \leq (q/m)|E|. \]

Integrating with respect to time from 0 to $t$, we obtain

\[ v - u \leq (q/m)|E|t, \]

as required.