Worked Example
The Speed of a Charged Particle in an Electromagnetic Field

A particle of mass m and charge q moves in a constant uniform electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. Gravity may be ignored. Write down an expression for the particle's energy and show that it is conserved. Prove that if the particle starts from $\mathbf{x}_0$ with speed u, and a time t later is at $\mathbf{x}$ with speed v, then

$$v^2 = u^2 + 2\mathbf{a} \,.\, \mathbf{s}$$

where $\mathbf{s} = \mathbf{x} - \mathbf{x}_0$ for a suitable constant vector $\mathbf{a}$. Show further that

$$v \leqslant u + (q/m)|\mathbf{E}|t.$$

Method 1: The potential energy corresponding to the electric field is $-q\mathbf{E}.\mathbf{x}$. However, there is no potential energy corresponding to the magnetic field, since the magnetic field does no work on the particle. Therefore the energy E is given by

$$E = \tfrac{1}{2}m|\dot{\mathbf{x}}|^2 - q\mathbf{E} \,.\, \mathbf{x}.$$

To show that it is conserved, differentiate with respect to time:

$$\begin{aligned} \dot{E} &= \tfrac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{x}} \,.\, \dot{\mathbf{x}}) - q\mathbf{E} \,.\, \dot{\mathbf{x}} \\ &= m\dot{\mathbf{x}} \,.\, \ddot{\mathbf{x}} - q\mathbf{E} \,.\, \dot{\mathbf{x}}. \end{aligned}$$

But the equation of motion for the particle is given by the Lorentz force,

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}), \qquad (*)$$

so

$$\dot{E} = q\dot{\mathbf{x}} \,.\, (\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) - q\mathbf{E} \,.\, \dot{\mathbf{x}} = 0,$$

as required.

Method 2: Starting from the equation of motion $(*)$, dot with $\dot{\mathbf{x}}$ and integrate:

$$\begin{aligned} & m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \\ \Longrightarrow \qquad & m\dot{\mathbf{x}} \,.\, \ddot{\mathbf{x}} = q\dot{\mathbf{x}} \,.\, \mathbf{E} \\ \Longrightarrow \qquad & \tfrac{1}{2}m\dot{\mathbf{x}} \,.\, \dot{\mathbf{x}} = q\mathbf{x} \,.\, \mathbf{E} + \text{const.}, \end{aligned}$$

so

$$\tfrac{1}{2}m|\dot{\mathbf{x}}|^2 - q\mathbf{E} \,.\, \mathbf{x} = E$$

is constant, as before.

Now, having used either method for the first part, use conservation of E between times 0 and t:

$$\begin{aligned} & \tfrac{1}{2}mv^2 - q\mathbf{E} \,.\, \mathbf{x} = \tfrac{1}{2}mu^2 - q\mathbf{E} \,.\, \mathbf{x}_0 \\ \Longrightarrow \qquad & v^2 = u^2 + \frac{2q}{m}\mathbf{E} \,.\, (\mathbf{x} - \mathbf{x}_0), \end{aligned}$$

which is of the required form with $\mathbf{a} = (q/m)\mathbf{E}$.

For the last part, write E in terms of $v = |\dot{\mathbf{x}}|$ as follows and differentiate:

$$\begin{aligned}
& & E &= \tfrac{1}{2}mv^2 - q\mathbf{E}\,.\,\mathbf{x} \\
&\implies & 0 &= mv\dot{v} - q\mathbf{E}\,.\,\dot{\mathbf{x}} \\
&\implies & mv\dot{v} &= q\mathbf{E}\,.\,\dot{\mathbf{x}} \\
& & &\leqslant q|\mathbf{E}||\dot{\mathbf{x}}| \\
& & &= q|\mathbf{E}|v \\
&\implies & \dot{v} &\leqslant (q/m)|\mathbf{E}|.
\end{aligned}$$

Integrating with respect to time from 0 to t, we obtain

$$v - u \leqslant (q/m)|\mathbf{E}|t,$$

as required.