Worked Example
Small Oscillations with One Degree of Freedom

A bead of mass m is threaded onto a smooth wire bent into the shape of the curve $y^2 = \frac{2}{3}lx$, $0 \leqslant x \leqslant 2l$, where (x, y) are Cartesian coordinates with y vertically upwards. A light spring of natural length l and spring constant $k = \frac{3}{2}mg/l$ is attached at one end to the bead and at the other to a smooth vertical track at $x = 2l$ along which it can slide, so that the spring remains horizontal. Show that there is a stable equilibrium of the bead at $x = y = \frac{2}{3}l$ and find the frequency of small oscillations about this point.

We will treat the system as having a single degree of freedom specified by y (although x would have been an equally good choice). The potential energy due to gravity is mgy, and the potential energy in the spring is

$$\tfrac{1}{2}k(l-x)^2 = \frac{3mg}{4l}\left(l - \frac{3}{2l}y^2\right)^2 = \frac{3mg}{4l}\left(l^2 - 3y^2 + \frac{9}{4l^2}y^4\right).$$

Therefore the total potential energy of the system is given by

$$V(y) = mgy + \frac{3mg}{4l}\left(-3y^2 + \frac{9}{4l^2}y^4\right)$$

(ignoring a constant term), so

$$V'(y) = mg + \frac{3mg}{4l}\left(-6y + \frac{9}{l^2}y^3\right)$$

and in particular,

$$V'(\tfrac{2}{3}l) = mg + \frac{3mg}{4l}(-4l + \tfrac{8}{3}l) = mg - 3mg + 2mg = 0.$$

So $y = \frac{2}{3}l$ is an equilibrium point; it is stable because

$$V''(\tfrac{2}{3}l) = \frac{3mg}{4l}\left(-6 + \frac{27}{l^2}(\tfrac{2}{3}l)^2\right) = \frac{9mg}{2l} > 0.$$

To find the frequency of small oscillations, we must first write the kinetic energy of the bead, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, in terms of $\dot{y}$ alone. Using

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2l}y^2\right) = 3y\dot{y}/l$$

we obtain $T = \frac{1}{2}m\dot{y}^2(1 + 9y^2/l^2)$; the total energy is therefore

$$E = \tfrac{1}{2}m\dot{y}^2\left(1 + \frac{9}{l^2}y^2\right) + V(y).$$

Differentiating with respect to time and recalling that E is constant,

$$\begin{aligned} m\dot{y}\ddot{y}\left(1+\frac{9}{l^2}y^2\right)+\tfrac{1}{2}m\dot{y}^2\left(\frac{18}{l^2}y\dot{y}\right)+V'(y)\dot{y}=0 \\ \implies \qquad m\ddot{y}\left(1+\frac{9}{l^2}y^2\right)+\frac{9m}{l^2}y\dot{y}^2+V'(y)=0. \end{aligned}$$

Let $y=\frac{2}{3}l+\varepsilon$, where ε is a small disturbance:

$$m\ddot{\varepsilon}\left(1+\frac{9}{l^2}(\tfrac{2}{3}l+\varepsilon)^2\right)+\frac{9m}{l^2}(\tfrac{2}{3}l+\varepsilon)\dot{\varepsilon}^2+V'(\tfrac{2}{3}l+\varepsilon)=0.$$

We can ignore terms of second order or higher (such as $\varepsilon\ddot{\varepsilon}$ and $\dot{\varepsilon}^2$). Expanding V' using a Taylor Series as $V'(\frac{2}{3}l+\varepsilon)=V'(\frac{2}{3}l)+\varepsilon V''(\frac{2}{3}l)+\cdots$ we obtain

$$\begin{aligned} m\ddot{\varepsilon}(1+4)+V'(\tfrac{2}{3}l)+\varepsilon V''(\tfrac{2}{3}l)=0 \\ \implies \qquad \ddot{\varepsilon}+\frac{9g}{10l}\varepsilon=0. \end{aligned}$$

The frequency of small oscillations is therefore $\sqrt{9g/(10l)}$. (Note that this is not equal to $\sqrt{V''(\frac{2}{3}l)/m}$, the value suggested by the "standard" result for truly one-dimensional motion.)