Dynamics: Example Sheet 1

1 Two cyclists start 40 km apart and head towards each other, each going at a steady rate of 20 km per hour. At the same time a fly that travels at a steady 30 km per hour starts from the nose of the southbound cyclist, flies to the nose of the northbound one, then back to the nose of the southbound one and so on. When the cyclists meet what distance has the fly covered? [*Attributed to John von Neumann, 1938.*]

2 Show that the trajectory of a ball thrown with speed V at an angle α to the horizontal is given by

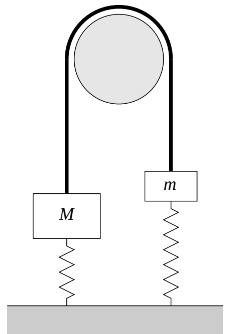
$$x = Vt\cos\alpha, \qquad y = Vt\sin\alpha - \tfrac{1}{2}gt^2.$$

Using polar coordinates (r, θ), establish the equivalent form

$$r = \frac{2V^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2\theta}.$$

If the ground slopes uphill with angle β, find the horizontal range of the ball. At what angle should the ball be thrown to maximize the range? [*No differentiation is needed.*]

3 Two masses m and M $(M > m)$, which are connected by a light, inextensible string passing over a light smooth pulley, are also connected to the ground by springs with spring constant k. When the masses are held level with each other, each spring is at its natural length. The masses are now allowed to move to their equilibrium positions: find the displacement of each spring due to gravity when the system is at rest. Furthermore, show that the natural frequency of small oscillations of the system is $\sqrt{2k/(M+m)}$, assuming that the string remains taut. Find the tension in the string during the oscillation: what is the maximum possible amplitude of oscillation that ensures the string remains taut?



4 Working in only one spatial dimension, the coordinate systems (x, t) and (x', t') of two frames of reference S and S' respectively are related by

$$x' = f(x,t), \qquad t' = t$$

for some function f. A particle follows a trajectory $x = x(t)$ in S, so that its trajectory in S' is $x' = f(x(t), t)$. Using the chain rule, show that its speed and acceleration in S' are given by

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \dot{x}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t},$$

$$\frac{\mathrm{d}^2x'}{\mathrm{d}t'^2} = \ddot{x}\frac{\partial f}{\partial x} + \dot{x}^2\frac{\partial^2 f}{\partial x^2} + 2\dot{x}\frac{\partial^2 f}{\partial x\,\partial t} + \frac{\partial^2 f}{\partial t^2}.$$

Supposing that S is an inertial frame, show that Newton's Second Law holds in S' for *all* trajectories $x(t)$ if and only if $f(x,t) = x + kt + c$ for some constants c and k. By what name is such a relationship between two frames known?

5 (i) Let (r, θ) be plane polar coordinates and let $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ be unit vectors in the direction of increasing r and increasing θ respectively. A particle moves in the plane with position vector $\mathbf{r} = r\hat{\mathbf{e}}_r$. Show that the velocity and acceleration are given by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta, \qquad \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta.$$

(ii) The particle now moves on the equiangular spiral

$$r = a\exp(\theta\cot\alpha),$$

where a and α are constants, with *constant* speed V. Show that

$$V = r\dot{\theta}\,\mathrm{cosec}\,\alpha$$

and hence that the particle's velocity is

$$\dot{\mathbf{r}} = V\cos\alpha\,\hat{\mathbf{e}}_r + V\sin\alpha\,\hat{\mathbf{e}}_\theta.$$

By differentiating this expression with respect to time, or otherwise, find also the (plane polar) components of the particle's acceleration $\ddot{\mathbf{r}}$ and hence show that $\dot{\mathbf{r}}\,.\,\ddot{\mathbf{r}} = 0$. Evaluate $|\ddot{\mathbf{r}}|$ in terms of r, V and α (but not $\dot{\theta}$).

(iii) Now suppose that the particle moves with variable speed $v(t)$ on the same spiral, but so that $\dot{\theta}$ takes a constant value ω. Show that the acceleration has magnitude v^2/r and is directed at an angle 2α to the position vector.

6 An undamped simple harmonic oscillator with natural frequency ω is initially at rest at the origin, but for $t \geqslant 0$ is subjected to forcing proportional to $\cos(\omega' t)$ where $\omega' = \omega + 2\varepsilon$ and $\varepsilon \ll \omega$. Show that the response of the oscillator is proportional to

$$\frac{\sin\big((\omega+\varepsilon)t\big)\sin\varepsilon t}{\varepsilon(\omega+\varepsilon)}$$

and sketch this response.

Explain qualitatively (without detailed calculation) why a *damped* harmonic oscillator cannot exhibit similar behaviour in the long term.

[*This phenomenon is known as "beats".*]

7 A particle of mass m and charge q moves in a uniform horizontal magnetic field $\mathbf{B}$ under the influence of gravity $\mathbf{g}$ acting vertically downwards. Show that the particle has a helical motion but with a constant horizontal drift, which you should find. An experimenter wishes to eliminate the drift by imposing a uniform electric field $\mathbf{E}$; what should the direction and magnitude of the field be? If the particle is a single electron, is this magnitude experimentally feasible? [*The mass and charge of an electron are approximately* $9{\cdot}1 \times 10^{-31}$ kg *and* $-1{\cdot}6 \times 10^{-19}$ C *respectively.*]

8 At time $t = 0$, an insect of mass m jumps from a point O on the ground with velocity $\mathbf{V}$, while a horizontal wind of velocity $\mathbf{U}$ is blowing. The gravitational acceleration is $\mathbf{g}$ and the air exerts a force on the insect equal to mk times the velocity of the wind *relative to* the insect.

(i) Show that the path of the insect is given by

$$\mathbf{x} = (\mathbf{U} + \mathbf{g}/k)t + k^{-1}(\mathbf{V} - \mathbf{U} - \mathbf{g}/k)(1 - \mathrm{e}^{-kt}).$$

(ii) In the case where the insect jumps vertically, find an equation satisfied by the time T that elapses before it returns to earth, and show that it will land at a distance

$$R = \frac{UT}{1 + g/kV}$$

from O, where $U = |\mathbf{U}|$, $V = |\mathbf{V}|$ and $g = |\mathbf{g}|$.

$*$ When $kV/g \ll 1$, show that

$$R \approx \frac{2kUV^2}{g^2}\left(1 - \frac{4}{3}\frac{kV}{g}\right).$$

9 A ball of mass m is projected vertically upwards with initial speed u in a resisting medium that produces a retardation force of magnitude kv^2, where v is the ball's speed. Show that when the ball returns to its point of projection, its final speed w satisfies

$$\frac{1}{w^2} = \frac{1}{u^2} + \frac{k}{mg}.$$

What has happened to the missing energy?

10 A raindrop of mass $3kV/g$ is falling vertically with speed $2V$ as it enters, at $t = 0$, a cloud which is falling vertically with *uniform* speed V. Whilst in the cloud, the drop acquires mass by condensation at a uniform rate k, and there is a resistance force of k times the relative speed. Let v be the speed of the drop relative to the cloud. Justify the assertion that the cloud defines an approximate inertial frame, and deduce the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg - kv,$$

where $m(t) = k(t + 3V/g)$. By changing the independent variable to m, or otherwise, show that $v = mg/3k$ and that the depth of the drop below the top of the cloud at time t is $Vt + \frac{1}{6}gt^2$.

11 A firework of initial mass m_0 is fired vertically upwards from the ground. The rate of burning of fuel $\mathrm{d}m/\mathrm{d}t$ is constant and equal to $-\alpha$, and the fuel is ejected at constant speed u relative to the firework. Show that the upward velocity of the firework at time t, where $0 < t < m_0/\alpha$, is

$$v(t) = -gt - u\log\Big(1 - \frac{\alpha t}{m_0}\Big),$$

and that this is positive provided $u > m_0 g/\alpha$.

Suppose now that nearly all of the firework consists of fuel, the mass of the containing shell being negligible. Show that the height attained by the shell when all of the fuel is burnt is

$$\frac{m_0}{\alpha}\Big(u - \frac{m_0 g}{2\alpha}\Big).$$

12 A satellite falls freely towards the Earth starting from rest at a distance R, much larger than the Earth's radius.

(i) Treating the Earth as a point mass M of negligible radius, use dimensional analysis to show that the time T taken by the satellite to reach the Earth is given by

$$T = CR^{3/2}(GM)^{-1/2},$$

where G is the gravitational constant and C is a dimensionless constant.

(ii) By integrating the equation of motion of the satellite, find an (implicit) equation for its distance from the Earth at time t, and deduce that $C = \pi/2\sqrt{2}$.

13 (i) A particle of mass m is released from rest at a height h above the ground. Neglecting air resistance and any variation in the gravitational acceleration g, show by dimensional analysis that the time taken to hit the ground is given by $T = C\sqrt{h/g}$, where C is a dimensionless constant.

(ii) Find the height $y(t)$ of the particle at time t, and deduce that $C = \sqrt{2}$.

(iii) Now suppose that air resistance produces a force equal to $-k$ times the velocity of the particle, so that T now depends on m, g, h and k. Use dimensional arguments to deduce that

$$T = \frac{m}{k}\,\tau(\lambda)$$

where $\lambda = k^2h/m^2g$ and $\tau(\lambda)$ is a dimensionless unknown function of the dimensionless quantity λ.

(iv) By integrating the equation of motion of the particle show that

$$\lambda = \tau + \mathrm{e}^{-\tau} - 1.$$

Sketch λ as a function of τ, and hence τ as a function of λ.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@cam.ac.uk`.