Mathematical Tripos Part IA Dr R. E. Hunt
Dynamics Lent 2007

Dynamics: Example Sheet 2

1 A bead of mass m is threaded on a smooth wire which is bent in the form of a *cycloid*, a plane curve defined parametrically by

$$x = a(\psi - \sin\psi), \qquad y = -a(1-\cos\psi), \qquad 0 \leqslant \psi \leqslant 2\pi,$$

where a is a positive constant and (x, y) are Cartesian coordinates with x horizontal and y vertical. The bead is released from rest at $\psi = 0$. Use conservation of energy to deduce that $\dot\psi^2 = g/a$ and find the period of the subsequent oscillations. Describe the motion.

* Now suppose that the bead is released from rest at $\psi = \psi_0$ where $0 < \psi_0 < \pi$. Show that the period of oscillations is given by

$$4\sqrt{\frac{a}{g}}\int_{\psi_0}^{\pi} \frac{\sin\frac{1}{2}\psi}{\sqrt{\cos^2\frac{1}{2}\psi_0 - \cos^2\frac{1}{2}\psi}}\,\mathrm{d}\psi$$

and hence, or otherwise, show that it is in fact independent of the value of ψ_0.

2 A particle of mass m lies at the origin on a smooth table, and is connected by one spring to the fixed point $(-a, 0)$ (in Cartesian coordinates in the plane of the table) and by a second spring to the fixed point $(a, 0)$. The natural length of each spring is l $(< a)$, the spring constant is k, and the mass of the springs is negligible. Find the potential energy if the particle is displaced slightly, (i) to $(x, 0)$ (where $x < a - l$); (ii) to $(0, y)$.

In each case find also the frequency of oscillation of the particle.

3 A particle of mass m and charge $-e$ moves in a uniform magnetic field $\mathbf{B}$ towards a charge q fixed at the origin. The force on the particle is therefore

$$\mathbf{F} = -\frac{qe}{4\pi\epsilon_0 r^3}\mathbf{r} - e\dot{\mathbf{r}}\times\mathbf{B},$$

where $\mathbf{r}(t)$ is its position and $r = |\mathbf{r}|$. Show that the quantity

$$E = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - \frac{qe}{4\pi\epsilon_0 r}$$

is a constant, and give a physical interpretation of each term.

Suppose that initially $\mathbf{r}\,.\,\mathbf{B} = 0$ and $\dot{\mathbf{r}}\,.\,\mathbf{B} = 0$. Show that in this case

$$m\mathbf{r}\times\dot{\mathbf{r}} - \tfrac{1}{2}er^2\mathbf{B}$$

is also constant.

4 In a hypothetical universe there is a sun which exerts no gravitational force. Instead it is surrounded by a force field which gives any other particle an acceleration

$$\ddot{\mathbf{x}} = \lambda \mathbf{x} \times \dot{\mathbf{x}},$$

where $\mathbf{x}(t)$ is the position vector of the particle relative to the sun and λ is a constant. Show that particles move in this field with constant speeds. By considering $|\mathbf{x}|^2$, or otherwise, show further that if the speed of the particle is v, and if it does not hit the sun, then its distance $r(t)$ from the sun is given by

$$r^2 = v^2\{(t - t_0)^2 + t_1^2\},$$

where t_0 and t_1 are constants.

5 A simple pendulum consists of a bob P of mass m attached to one end of a rigid massless rod of length l. The other end of the rod is pivoted at a fixed point O. A massless spring of natural length $\frac{1}{2}l$ and modulus of elasticity mg is also attached to the bob and connects it to a fixed point Q a distance l vertically above O. Let θ be the angle between OQ and OP. Show that the equilibrium positions are at $\theta = \frac{\pi}{3}$ and $\theta = \pi$, and (where appropriate) find the period of small oscillations about the equilibrium.

6 A particle of charge q, mass m and position vector $\mathbf{r}$ moves in the electrostatic field of a charge Q fixed at the origin (where $Qq > 0$) *and* in an attractive force field $\mathbf{F} = -p\mathbf{r}/|\mathbf{r}|^4$. If the particle starts a distance r_0 from the origin, what is the minimum initial speed that must be imparted to it in order to escape to infinity? What happens if $r_0 > 4\pi\epsilon_0 p/(Qq)$?

7 A man of mass m stands in a lift cage of mass M which is attached to a counterweight of mass $M+m$ by a light, inextensible cable passing over a light, smooth pulley. Initially the lift is at rest. The man jumps, applying to the floor an impulse I that would have raised him to a height h had he jumped from the ground. Show that his greatest distance above the floor of the lift is

$$\frac{2(M+m)}{2M+m}h.$$

* What would the answer have been had the lift initially been descending with speed I/m?

8 Hailstones falling at an angle of 30° to the vertical strike the frozen surface of the river Cam, and rebound at an angle 60° to the vertical and to a height h. Ignoring any friction show that the coefficient of restitution is $\frac{1}{3}$ and that the speed with which the hailstones strike the surface is $2\sqrt{6gh}$.

9 *Superballs* is an American toy made by the "Wham-O Mfg. Co." of California, USA. It consists of a number of almost perfectly elastic balls of various sizes. The instructions suggest that one place a light ball on top of a heavy one, and then drop the combination from rest at a height h. Apparently with the right combination one can arrange for the heavier ball to come to rest after the bounce. What ratio of masses ensures this?

However, children soon find a much more interesting game. Show that with an appropriate choice of balls the lighter one can rise almost to $9h$. Show also that, using three balls, the lightest one can rise almost to $49h$. ("Wham-O" expect to sell plenty of spare balls!)

10 A bowling ball of mass m is rolled at speed v towards a stationary ball of mass $2m$. The collision, which has coefficient of restitution $e = \frac{1}{3}$, is oblique, so that the normal to the plane of contact between the two balls at the point of collision makes an angle of $30°$ with the velocity vector of the first ball. Working in the centre of mass frame, find the speeds of the balls after the collision. Also calculate the combined kinetic energy of the balls in the centre of mass frame before and after the collision, and show that it is reduced to one-third of its original value.

11 A particle moves in a plane under the influence of a repulsive central force which obeys an inverse-cube law. Show from first principles that $\dot{r}^2 + k/r^2$ is constant, where r is the distance of the particle from the centre of force and k is a positive constant.

12 (i) A particle P of mass m slides on a smooth horizontal table. It is attached by an inextensible string of length l which passes through a small smooth hole in the table at position O to a second mass M, which hangs vertically below O. The motion is such that the string remains taut at all times. If the polar coordinates of P relative to O are $\big(r(t), \theta(t)\big)$, show that $r^2\dot{\theta} = h$, a constant, and that

$$(m + M)\ddot{r} = -Mg + \frac{mh^2}{r^3}.$$

Deduce an energy conservation equation for the system.

(ii) Show that a possible solution of the system is $r = a$, $\dot{\theta} = \omega$ where a and ω are constants, provided that $Mg = ma\omega^2$. Let $r = a + \rho(t)$, where $|\rho| \ll a$. Show that with a linear approximation for small ρ, ρ executes simple harmonic motion, and find the frequency of the oscillations. Describe this motion physically. [*You may ignore any change in the angular momentum h.*]

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@cam.ac.uk`.