Mathematical Tripos Part IA | Dr R. E. Hunt
Dynamics | Lent 2007

Dynamics: Example Sheet 3

1 A particle moves in the gravitational field of the Sun and is influenced by radiation pressure. The particle has unit mass, speed v and polar coordinates (r, θ) in the plane of motion with the centre of the Sun as origin. The forces acting on the particle can be taken to be μ/r^2 towards the origin and kv opposing the motion, where μ and k are constants. Establish the equations

$$r^2\dot{\theta} = h\mathrm{e}^{-kt}, \qquad \mu r = h^2\mathrm{e}^{-2kt} - r^3(\ddot{r} + k\dot{r}),$$

where h is a constant.

Show that when $k = 0$, a circular orbit of radius a exists for any value of a, and find its angular frequency ω in terms of a and μ. When k is small (more precisely, when $k/\omega \ll 1$), the radius r changes very slowly, so that $\dot{r}$ and $\ddot{r}$ may be neglected compared to other terms: verify that in this situation an approximate solution is

$$r = a\mathrm{e}^{-2kt}, \qquad \dot{\theta} = \omega\mathrm{e}^{3kt}.$$

Give a brief qualitative description of the behaviour of this solution for $t > 0$. Does the speed of the particle increase or decrease?

2 A particle moves in a circular orbit of radius R under the influence of an attractive central force proportional to $r^{-2}\exp(-(r/a)^2)$, where a is a constant. Show that the orbit is stable or unstable to small perturbations according as $R < a/\sqrt{2}$ or $R > a/\sqrt{2}$ respectively.

$*$ What can be said about the stability if $R = a/\sqrt{2}\,$?

3 A frame of reference S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S. The two frames have a common origin O. Prove that

$$\left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_S = \left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}),$$

where $\mathbf{x}$ is the position vector of a point P measured from the origin, and give a clear explanation of the notation used.

A square hoop $ABCD$ is made of fine smooth wire and has side length $2a$. The hoop is horizontal and rotating with constant angular speed ω about a vertical axis through A. A small bead which can slide on the wire is initially at rest at the midpoint of the side BC. Choose axes fixed relative to the hoop, and let y be the distance of the bead from the vertex B on the side BC. Write down the position vector of the bead in the rotating frame.

Using the expression for the acceleration in S given above, show that

$$\ddot{y} - \omega^2 y = 0.$$

Hence show that the time which the bead takes to reach a corner of the hoop is $\omega^{-1}\cosh^{-1} 2$.

4 A fast train is travelling at speed V due north-east at latitude λ on the surface of the Earth, which rotates with constant angular velocity $\boldsymbol{\omega}$. A plumbline is attached to the ceiling inside one of the carriages and hangs down freely. Neglecting terms of order $|\boldsymbol{\omega}|^2$, find the apparent acceleration due to gravity and calculate its horizontal and vertical components. Show that the plumbline will be inclined at an angle to the vertical given approximately by $(2|\boldsymbol{\omega}|V/g)\sin\lambda$.

5 A particle of mass m is projected from a point Q with velocity $\mathbf{v}$ in a frame which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame. The particle is subject to a gravitational force $m\mathbf{g}$ which is constant in the rotating frame. Using the vector equation of motion and neglecting terms of order $|\boldsymbol{\omega}|^2$, show that the particle's position vector measured in the rotating frame is given by

$$\mathbf{q} + \mathbf{v}t + (\tfrac{1}{2}\mathbf{g} - \boldsymbol{\omega}\times\mathbf{v})t^2 + \tfrac{1}{3}\mathbf{g}\times\boldsymbol{\omega}t^3$$

at time t, where $\mathbf{q}$ is the position vector of Q. [*There is no need to take components.*]

Suppose that the particle is projected from sea level on the Earth at latitude λ in the Northern hemisphere, with initial velocity $\mathbf{v}$ at an angle $\pi/4$ from the upward vertical in a Northward direction. Show that when the particle returns to sea level, it has been deflected to the East by an amount approximately equal to

$$\frac{\sqrt{2}\,\omega|\mathbf{v}|^3}{3g^2}(3\sin\lambda - \cos\lambda),$$

where ω is the angular speed of the Earth.

Evaluate the approximate size of this deflection at latitude 52° N for $|\mathbf{v}| = 10\,\mathrm{m/s}$.

6 A river at latitude λ flows with a speed of 3 km per hour, and a typical bend has a radius of curvature R. Roughly how large does R have to be so that the Coriolis force from the Earth's rotation is greater than the centrifugal force produced by the flow of the river?

Explain why a large river in the Northern hemisphere undermines the base of its right bank, while a small river with abrupt bends of small radius undermines either the left or the right bank, whichever is outward from the bend.

7 A solid sphere of radius a has a non-uniform density distribution proportional to $1-r/a$ where r is the radial coordinate. A particle of unit mass is positioned within the sphere at a distance x from the centre. Find the gravitational potential energy and hence, or otherwise, the force on the particle. Show that the force is identical to that which would be produced by only that part of the sphere lying in $r < x$.

8 For circular and parabolic orbits having the same angular momentum in an attractive $1/r$ potential, show that the perihelion distance $r_{\min}$ of the parabola is half the radius of the circle.

No longer assuming equality of angular momentum, show further than the speed of a particle at any point of a parabolic orbit is $\sqrt{2}$ times the speed in a circular orbit passing through the same point.

9 A particle A of mass m is attracted by the gravitational force of a second particle B which is fixed at the origin. Initially, A is very far from B and has velocity $\mathbf{V}$ directed along a straight line which passes at a distance l from B. The shortest distance between A's trajectory and B is d. Deduce the mass of B in terms of the given quantities and the gravitational constant G. [*There is no need to find the eccentricity of the orbit.*]

10 A particle P of unit mass moves in a plane under a central force

$$f(u) = -\lambda u^3 - \mu u^2,$$

where $u = 1/r$ and λ, μ are positive constants. If P is projected with speed V from the point $r = r_0$, $\theta = 0$ in the direction perpendicular to OP, find the equation of the orbit under the assumptions

$$\lambda < V^2 r_0^2 < 2\mu r_0 + \lambda.$$

Explain the significance of these inequalities.

Show that between consecutive apsides (points of greatest or least distance) the radius vector turns through an angle

$$\pi\big(1 - \lambda/(V^2 r_0^2)\big)^{-1/2}.$$

∗ Under what condition is the orbit a closed curve?

11 A particle P of mass m moves under the influence of a central force of magnitude mkr^{-3} directed towards a fixed point O. Initially $r = a$ and P has velocity V perpendicular to OP, where $V^2 < k/a^2$. Prove that P spirals in towards O and reaches O in a time

$$T = \frac{a^2}{\sqrt{k - a^2V^2}}.$$

12 Consider a particle of unit mass moving in a central force field with potential $V(r) = -\alpha/r$ where $\alpha > 0$. Show that the angular momentum $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ about the origin is constant, and deduce that the orbit lies in a plane containing the origin. Let $\mathbf{L} = \dot{\mathbf{r}} \times \mathbf{h} - \alpha\mathbf{r}/r$: show that $\mathrm{d}\mathbf{L}/\mathrm{d}t = \mathbf{0}$ and deduce that $\mathbf{L}$ is constant.

∗ By evaluating $\mathbf{L}$ at perihelion, show that $\mathbf{L}$ is directed along the major axis from the focus to the perihelion, and that $|\mathbf{L}| = \alpha e$ where e is the eccentricity of the orbit. [*Hint: use* $r = (h^2/\alpha)/(1 + e\cos\theta)$.]

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@cam.ac.uk`.