Dynamics: Example Sheet 4

1 A particle of mass $3m$ at rest is hit by a particle of mass m moving at velocity $\mathbf{v}$. The collision produces n fragments of equal mass, each with equal energy relative to the centre of mass. Assuming that energy is conserved during the collision, show that the fragments spread out on a sphere whose centre moves with velocity $\frac{1}{4}\mathbf{v}$ and whose radius expands with speed $\frac{\sqrt{3}}{4}|\mathbf{v}|$.

2 A system of particles with masses m_i and position vectors $\mathbf{x}_i$, $i = 1, \ldots, n$, moves under its own mutual gravitational attraction alone. Show that a possible solution of the equations of motion is given by $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the $\mathbf{a}_i$ are constant vectors, if the $\mathbf{a}_i$ satisfy

$$\mathbf{a}_i = \tfrac{9}{2}G\sum_{j\neq i}\frac{m_j(\mathbf{a}_i - \mathbf{a}_j)}{|\mathbf{a}_i - \mathbf{a}_j|^3}.$$

Show that for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other fixed point?

3 Two gravitating particles with masses m_1 and m_2 start from rest a large distance apart. They are allowed to fall freely towards one another. Prove that their centre of mass is fixed, and show further that when they are a distance r apart, their relative speed is $\sqrt{2G(m_1 + m_2)/r}$.

* If the particles are given equal and opposite impulses I when they are a distance a apart, such that each impulse is perpendicular to the direction of motion, find an expression for their minimum distance of separation, d, in the subsequent motion. Show that if $2G(m_1 + m_2)\mu^2 \gg I^2 a$ then $d \approx I^2a^2(m_1 + m_2)/(2Gm_1^2m_2^2)$, where μ is the reduced mass.

4 State the parallel axis and perpendicular axis theorems.

(i) Find the moment of inertia of a uniform solid cylinder of mass M, length l and radius a about its axis; about a perpendicular axis through the centre of mass; and about an axis which is parallel to the original axis and tangent to the surface.

(ii) Find the moment of inertia of a solid circular cone of mass M, height h and base radius a about its axis, and also about a perpendicular axis through its apex.

* (iii) A flat uniform circular disc of radius a has a hole in it of radius b whose centre is at a distance c from the centre of the disc, where $c < a - b$. The disc is free to oscillate in a vertical plane about a smooth horizontal circular rod of radius b passing through the hole. Find the moment of inertia of the disc about the centre of the hole and hence show that the length of the equivalent simple pendulum is $c + (a^4 - b^4)/(2a^2c)$.

5 A uniform circular cylinder of mass M and radius a is free to turn about its axis which is horizontal. A thin uniform cylindrical shell of mass $\frac{1}{2}M$ and radius a is fitted over the cylinder. At time $t = 0$ the angular velocity of the cylinder is Ω, while the shell is at rest. The shell exerts a frictional couple on the cylinder of magnitude $k(\omega - \varpi)$, where $\omega(t)$ and $\varpi(t)$ are the angular velocities of the cylinder and shell respectively at time t about the axis. Prove that $\omega(t) = \frac{1}{2}\Omega\{1 + \exp(-4kt/(Ma^2))\}$, and find the corresponding expression for $\varpi(t)$.

6 A yo-yo consists of two uniform heavy discs, each of mass M and radius R, connected by a light axle of radius a around which one end of a string is wound. The other end of the string is held fixed. Assuming that the unwound part of the string is approximately vertical, how fast does the yo-yo accelerate downwards?

The yo-yo is released from rest. Show by direct calculation that when its centre of mass has fallen a distance h, its kinetic energy is $2Mgh$, and deduce that the total energy is conserved.

7 A marble of mass M and radius a is launched along a rough horizontal table with speed V and with backspin, so that its initial angular velocity is $-\Omega$. So long as the marble is slipping along the table, the coefficient of (sliding) friction is μ. Show that if $\Omega < 5V/(2a)$ then the marble continues rolling in the same direction; but if $\Omega > 5V/(2a)$ then it reverses direction and passes its initial position again. What range of Ω ensures that it is rolling (rather than slipping) when it passes its initial position? What is its speed at that point?

What happens if $\Omega = 5V/(2a)$?

8 Find the fixed point of the system

$$\dot{x} = -x + y, \qquad \dot{y} = -x - y$$

and classify it. By transforming from Cartesian coordinates to polar coordinates (r, θ), verify your classification and sketch the phase portrait.

9 For a particle of unit mass moving in one dimension in the following potentials, find the equilibrium points and classify them. In each case, sketch the phase diagram and (where relevant) find the equations of the separatrices. Describe the possible types of motion qualitatively.

(i) $V(x) = x^2$;

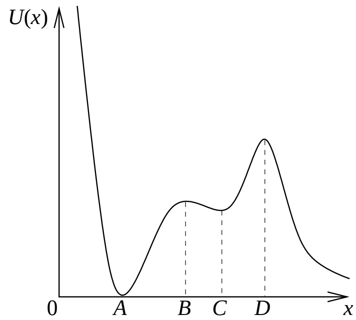
(ii) $V(x) = -x^2$;

(iii) $V(x) = x^2 \exp(-x^2)$;

(iv) $V(x) = x^{-4} - 2x^{-2}$, where the motion is confined to $x > 0$.

In cases (i) and (iv), find also the period of small oscillations about the stable fixed point.

10 Sketch a qualitative phase diagram corresponding to the motion of a particle moving in one dimension under the potential $U(x)$ shown in the diagram. Identify the fixed points and separatrices.



11 The numbers $a(t)$ of aardvarks and $b(t)$ of buffaloes in an enclosed reserve satisfy

$$\dot{a} = a - 2ab, \qquad \dot{b} = 2b - ab.$$

How might the individual terms in these equations be interpreted? Find the critical points and determine their stability types. Hence sketch the phase portrait.

∗ Solve the equation

$$\frac{\mathrm{d}a}{\mathrm{d}b} = \frac{a(1-2b)}{b(2-a)}$$

and thereby verify your stability result for the non-zero critical point by making a series expansion of the solution in a neighbourhood of the point.

12 A dynamical system described by variables (q, p) is governed by the equations

$$\dot{q} = p + \sin q,$$
$$\dot{p} = -p\cos q.$$

Show that the quantity $H = \frac{1}{2}p^2 + p\sin q$ is conserved by this system.

Find the fixed points and determine their stability. Sketch the phase diagram (with q along the horizontal axis and p along the vertical axis), showing clearly where libration and where rotation occur. Show that the period of librating motion is

$$T = 4\int_0^{\cos^{-1}\sqrt{2|H|}} \frac{\mathrm{d}\theta}{\sqrt{\cos^2\theta - 2|H|}}.$$

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@cam.ac.uk`.