Summary of Results from Previous Courses

**Grad, Div, Curl and the Laplacian in Cartesian Coordinates**

In Cartesian coordinates, \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \). For a scalar field \( \Phi(x) \) and a vector field \( \mathbf{F}(x) = (F_1, F_2, F_3) \), we define:

**Gradient** \( \nabla \Phi = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right) \) (“grad Phi”)

**Divergence** \( \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \) (“div \( \mathbf{F} \)”)

**Curl** \( \nabla \times \mathbf{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \) (“curl \( \mathbf{F} \)”)

**Laplacian** \( \nabla^2 \Phi = \nabla \cdot (\nabla \Phi) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \) (“del-squared Phi”)

The normal to a surface \( \Phi(x) = \text{constant} \) is parallel to \( \nabla \Phi \).

**Grad, Div and the Laplacian in Polar Coordinates**

**Cylindrical Polars** \((r, \theta, z)\)

When the components \((F_1, F_2, F_3)\) of \( \mathbf{F} \) are measured in cylindrical polar coordinates,

\[
\nabla \Phi = \left( \frac{\partial \Phi}{\partial r}, \frac{1}{r} \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial z} \right)
\]

\[
\nabla \cdot \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_1) + \frac{1}{r} \frac{\partial F_2}{\partial \theta} + \frac{\partial F_3}{\partial z}
\]

\[
\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}
\]

**Note:** the formulae for plane polar coordinates \((r, \theta)\) are obtained by setting \( \frac{\partial}{\partial z} = 0 \).

**Spherical Polars** \((r, \theta, \phi)\)

When the components \((F_1, F_2, F_3)\) of \( \mathbf{F} \) are measured in spherical polar coordinates,

\[
\nabla \Phi = \left( \frac{\partial \Phi}{\partial r}, \frac{1}{r} \frac{\partial \Phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \right)
\]

\[
\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_1) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (F_2 \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial F_3}{\partial \phi}
\]

\[
\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2}
\]
Divergence and Stokes’ Theorems

Divergence Theorem in 3D
\[ \iiint_V \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \]
where the surface \( S \) encloses a volume \( V \) and \( \mathbf{n} \) is its outward-pointing normal.

Divergence Theorem in 2D
\[ \iint_S \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \, dx \, dy = \oint_C (f \, dy - g \, dx) \]
where \( S \) is a plane region enclosed by a contour \( C \) traversed anti-clockwise. We can also write the right-hand side as \( \oint_C \mathbf{F} \cdot \mathbf{n} \, dl \) where \( \mathbf{F} = (f, g) \) and \( \mathbf{n} \) is the outward-pointing normal on \( C \).

Stokes’ Theorem
\[ \iint_S \left( \nabla \times \mathbf{F} \right) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot \mathbf{d}l \]
where the open surface \( S \) is bounded by a contour \( C \), \( \mathbf{n} \) is the normal to \( S \) and \( \mathbf{d}l \) is a line element taken anti-clockwise around \( C \).

Sturm–Liouville Theory

A Sturm–Liouville equation in self-adjoint form
\[-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \lambda w(x)y\]
in an interval \( a < x < b \), where neither \( p(x) \) nor \( w(x) \) vanish in the interval, and with “appropriate” boundary conditions, has non-zero solutions only for certain values of \( \lambda \), namely the eigenvalues \( \lambda_i \). The corresponding solutions \( y_i(x) \) (the eigenfunctions) are orthogonal for distinct eigenvalues: \( \int_a^b w_i y_j \, dx = 0, i \neq j \).

Vectors and Matrices

Vector identities:
\[ |\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} \]
\[ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \]
\[ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \]
\[ \nabla (\Phi \Psi) = \Phi \nabla \Psi + \Psi \nabla \Phi \]
\[ \nabla (\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} \]
\[ \nabla \cdot (\Phi \mathbf{u}) = \Phi \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \Phi \]
\[ \nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) \]
\[ \nabla \times (\Phi \mathbf{u}) = \Phi \nabla \times \mathbf{u} + \nabla \Phi \times \mathbf{u} \]
\[ \nabla \times (\mathbf{u} \times \mathbf{v}) = (\nabla \cdot \mathbf{v})\mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v} - (\nabla \cdot \mathbf{u})\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u} \]
\[ \nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \]
A matrix $A$ is orthogonal if $A^T A = A A^T = I$ where $I$ is the identity matrix and $A^T$ is the transpose of $A$. This is true if and only if the columns of $A$ are mutually orthogonal unit vectors; similarly for the rows. Then $A^{-1} = A^T$. In 3D, an orthogonal matrix is either a rotation, a reflection, or a combination of the two.

$x$ is an eigenvector of a symmetric matrix $A$ with eigenvalue $\lambda$ if $Ax = \lambda x$. The eigenvalues can be found by solving the equation $\det(A - \lambda I) = 0$. The three unit eigenvectors are orthogonal (or in the case of repeated eigenvalues, can be chosen to be so). The eigenvalues are also given by the stationary values of $a^T A a$ over all possible vectors $a$ (or equivalently, the eigenvalues are given by the stationary values of $a^T A a$ subject to the constraint $a^T a = 1$).

The determinant of a matrix is unchanged by adding a multiple of one row to a different row, or by adding a multiple of one column to a different column. Swapping two rows changes the sign of the determinant, as does swapping two columns. Multiplying a row, or a column, by a constant factor $\alpha$ multiplies the determinant by $\alpha$. If two rows, or columns, are the same, then the determinant is zero. For any square matrices $A$ and $B$, $\det(A^T) = \det(A)$ and $\det(AB) = \det(A) \det(B)$.

**Fourier Series**

Any (well-behaved) function $f(x)$ with period $L$ may be represented as the infinite sum

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{2n\pi x}{L} + B_n \sin \frac{2n\pi x}{L} \right)$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x) \, dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} \, dx, \quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} \, dx.$$ 

A function $f(x)$ which is defined only in the region $0 \leq x \leq L$ may be represented as a full Fourier Series as above by first turning it into a periodic function with period $L$; or may alternatively be represented either by a Fourier cosine series, in which only the cosine terms appear, or by a Fourier sine series, in which only the sine terms appear. For a cosine series,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x) \, dx \quad \text{and} \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx.$$ 

For a sine series,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$ 

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**Legendre Polynomials**

Legendre’s equation for $P(x)$ is

$$ \frac{d}{dx} \left( (1-x^2) \frac{dP}{dx} \right) + \lambda P = 0. $$

There are regular singular points at $x = \pm 1$. A series solution may be sought about $x = 0$; but the resulting series is ill-behaved (specifically, $P$ is singular at $x = \pm 1$) except when $\lambda = n(n + 1)$ for some non-negative integer $n$. Then the series terminates after a finite number of terms, and the solution is the Legendre polynomial $P_n(x)$ of degree $n$. $P_n(x)$ is an even/odd function of $x$ (i.e., contains only even/odd powers of $x$) when $n$ is even/odd respectively. It is normalised so that $P_n(1) = 1$ (and therefore $P_n(-1) = (-1)^n$). Legendre polynomials are orthogonal:

$$ \int_{-1}^{1} P_m(x)P_n(x) \, dx = \begin{cases} 0 & m \neq n, \\ \frac{2}{2n+1} & m = n. \end{cases} $$

They can be found explicitly using Rodrigues’ formula

$$ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left\{ (x^2 - 1)^n \right\}. $$

**Taylor’s Theorem (complex version)**

Any smooth complex function can be expressed as a power series about $z = z_0$ in the form

$$ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n $$

where $a_n = f^{(n)}(z_0)/n!$.

**Fourier Transforms**

For suitable functions $f(x)$, the Fourier Transform is defined by

$$ \tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx, $$

and the inversion formula is

$$ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, dk. $$

The Fourier Transform of $f'(x)$ is $ik \tilde{f}(k)$. The Fourier Transform of $f(x-a)$ for constant $a$ is $e^{-ika} \tilde{f}(k)$. The convolution $h = f * g$, defined by

$$ h(y) = \int_{-\infty}^{\infty} f(x) g(y-x) \, dx, $$

satisfies $\tilde{h}(k) = \tilde{f}(k) \tilde{g}(k)$.

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