# Summary of Results from Previous Courses

## *Grad, Div, Curl and the Laplacian in Cartesian Coordinates*

In Cartesian coordinates, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. For a scalar field $\Phi(\mathbf{x})$ and a vector field $\mathbf{F}(\mathbf{x}) = (F_1, F_2, F_3)$, we define:

**Gradient** $\quad \nabla\Phi = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right)$ (“grad Phi”)

**Divergence** $\quad \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ (“div **F**”)

**Curl** $\quad \nabla\times\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ (“curl **F**”)

**Laplacian** $\quad \nabla^2\Phi = \nabla\cdot(\nabla\Phi) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$ (“del-squared Phi”)

The normal to a surface $\Phi(\mathbf{x}) =$ constant is parallel to $\nabla\Phi$.

## *Grad, Div and the Laplacian in Polar Coordinates*

**Cylindrical Polars** $(r, \theta, z)$

When the components $(F_1, F_2, F_3)$ of $\mathbf{F}$ are measured in cylindrical polar coordinates,

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial z}\right)$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{\partial\theta} + \frac{\partial F_3}{\partial z}$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

**Note**: the formulae for plane polar coordinates $(r, \theta)$ are obtained by setting $\frac{\partial}{\partial z} = 0$.

**Spherical Polars** $(r, \theta, \phi)$

When the components $(F_1, F_2, F_3)$ of $\mathbf{F}$ are measured in spherical polar coordinates,

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial\phi}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

## *Divergence and Stokes' Theorems*

**Divergence Theorem in 3D** $$\iiint_V \nabla \cdot \mathbf{F}\, \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$$

where the surface $S$ encloses a volume $V$ and $\mathbf{n}$ is its outward-pointing normal.

**Divergence Theorem in 2D** $$\iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y = \oint_C (f\, \mathrm{d}y - g\, \mathrm{d}x)$$

where $S$ is a plane region enclosed by a contour $C$ traversed anti-clockwise. We can also write the right-hand side as $\oint_C \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}l$ where $\mathbf{F} = (f, g)$ and $\mathbf{n}$ is the outward-pointing normal on $C$.

**Stokes' Theorem** $$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}S = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}$$

where the open surface $S$ is bounded by a contour $C$, $\mathbf{n}$ is the normal to $S$ and $\mathrm{d}\mathbf{l}$ is a line element taken anti-clockwise around $C$.

## *Sturm–Liouville Theory*

A Sturm–Liouville equation in self-adjoint form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \lambda w(x)y$$

in an interval $a < x < b$, where neither $p(x)$ nor $w(x)$ vanish in the interval, and with "appropriate" boundary conditions, has non-zero solutions only for certain values of $\lambda$, namely the eigenvalues $\lambda_i$. The corresponding solutions $y_i(x)$ (the eigenfunctions) are orthogonal for distinct eigenvalues: $\int_a^b w y_i y_j\, \mathrm{d}x = 0$, $i \neq j$.

## *Vectors and Matrices*

Vector identities:

$$\begin{aligned}
|\mathbf{u}|^2 &= \mathbf{u} \cdot \mathbf{u} \\
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \\
\nabla(\Phi\Psi) &= \Phi\nabla\Psi + \Psi\nabla\Phi \\
\nabla(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \times (\nabla \times \mathbf{v}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} \\
\nabla \cdot (\Phi\mathbf{u}) &= \Phi\, \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\Phi \\
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) \\
\nabla \times (\Phi\mathbf{u}) &= \Phi\, \nabla \times \mathbf{u} + \nabla\Phi \times \mathbf{u} \\
\nabla \times (\mathbf{u} \times \mathbf{v}) &= (\nabla \cdot \mathbf{v})\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{v} - (\nabla \cdot \mathbf{u})\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{u} \\
\nabla^2\mathbf{u} &= \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})
\end{aligned}$$

A matrix $A$ is orthogonal if $A^TA = AA^T = I$ where $I$ is the identity matrix and $A^T$ is the transpose of $A$. This is true if and only if the columns of $A$ are mutually orthogonal unit vectors; similarly for the rows. Then $A^{-1} = A^T$. In 3D, an orthogonal matrix is either a rotation, a reflection, or a combination of the two.

$\mathbf{x}$ is an eigenvector of a symmetric matrix $A$ with eigenvalue $\lambda$ if $A\mathbf{x} = \lambda\mathbf{x}$. The eigenvalues can be found by solving the equation $\det(A - \lambda I) = 0$. The three unit eigenvectors are orthogonal (or in the case of repeated eigenvalues, can be chosen to be so). The eigenvalues are also given by the stationary values of $\mathbf{a}^TA\mathbf{a}/\mathbf{a}^T\mathbf{a}$ over all possible vectors $\mathbf{a}$ (or equivalently, the eigenvalues are given by the stationary values of $\mathbf{a}^TA\mathbf{a}$ subject to the constraint $\mathbf{a}^T\mathbf{a} = 1$).

The determinant of a matrix is unchanged by adding a multiple of one row to a different row, or by adding a multiple of one column to a different column. Swapping two rows changes the sign of the determinant, as does swapping two columns. Multiplying a row, or a column, by a constant factor $\alpha$ multiplies the determinant by $\alpha$. If two rows, or columns, are the same, then the determinant is zero. For any square matrices $A$ and $B$, $\det A^T = \det A$ and $\det AB = \det A \det B$.

## *Fourier Series*

Any (well-behaved) function $f(x)$ with period $L$ may be represented as the infinite sum

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{2n\pi x}{L} + B_n \sin \frac{2n\pi x}{L} \right)$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x)\,\mathrm{d}x, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\,\mathrm{d}x, \quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\,\mathrm{d}x.$$

A function $f(x)$ which is defined only in the region $0 \leqslant x \leqslant L$ may be represented as a full Fourier Series as above by first turning it into a periodic function with period $L$; or may alternatively be represented either by a Fourier cosine series, in which only the cosine terms appear, or by a Fourier sine series, in which only the sine terms appear. For a cosine series,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x)\,\mathrm{d}x \qquad \text{and} \qquad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,\mathrm{d}x.$$

For a sine series,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\,\mathrm{d}x.$$

## Legendre Polynomials

Legendre's equation for $P(x)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right)+\lambda P=0.$$

There are regular singular points at $x=\pm 1$. A series solution may be sought about $x=0$; but the resulting series is ill-behaved (specifically, $P$ is singular at $x=\pm 1$) *except* when $\lambda=n(n+1)$ for some non-negative integer $n$. Then the series terminates after a finite number of terms, and the solution is the Legendre polynomial $P_n(x)$ of degree $n$. $P_n(x)$ is an even/odd function of $x$ (i.e., contains only even/odd powers of $x$) when $n$ is even/odd respectively. It is normalised so that $P_n(1)=1$ (and therefore $P_n(-1)=(-1)^n$). Legendre polynomials are orthogonal:

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x=\begin{cases}0 & m\neq n,\\ \frac{2}{2n+1} & m=n.\end{cases}$$

They can be found explicitly using Rodrigues' formula

$$P_n(x)=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\{(x^2-1)^n\}.$$

## Taylor's Theorem (complex version)

Any smooth complex function can be expressed as a power series about $z=z_0$ in the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

where $a_n=f^{(n)}(z_0)/n!$.

## Fourier Transforms

For suitable functions $f(x)$, the Fourier Transform is defined by

$$\widetilde{f}(k)=\int_{-\infty}^{\infty}f(x)e^{-ikx}\mathrm{d}x,$$

and the inversion formula is

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{f}(k)e^{ikx}\mathrm{d}k.$$

The Fourier Transform of $f'(x)$ is $ik\widetilde{f}(k)$. The Fourier Transform of $f(x-a)$ for constant $a$ is $e^{-ika}\widetilde{f}(k)$. The convolution $h=f*g$, defined by

$$h(y)=\int_{-\infty}^{\infty}f(x)g(y-x)\,\mathrm{d}x,$$

satisfies $\widetilde{h}(k)=\widetilde{f}(k)\widetilde{g}(k)$.