Worked Example
Constrained Maximization

A cuboid is inscribed in an ellipse with semi-axes a, b and c. What is its maximum volume?

We must find values of x, y and z which maximize the cuboid's volume $f(x,y,z) \equiv 8xyz$ subject to the constraint

$$g(x,y,z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

which ensures that the vertices of the cuboid lie on the surface of the ellipse. We introduce an undetermined multiplier λ and consider the three components of the equation $\nabla(f - \lambda g) = \mathbf{0}$:

$$\begin{aligned} 8yz - \frac{2\lambda x}{a^2} &= 0, \\ 8xz - \frac{2\lambda y}{b^2} &= 0, \\ 8xy - \frac{2\lambda z}{c^2} &= 0. \end{aligned}$$

Multiplying these equations by x, y and z respectively, we see that

$$8xyz = 2\lambda \frac{x^2}{a^2} = 2\lambda \frac{y^2}{b^2} = 2\lambda \frac{z^2}{c^2}.$$

Hence either

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

or $\lambda = 0$. The latter possibility would imply that the volume is zero, which is clearly a minimum rather than the maximum which we seek, so we exclude this from now on. Remembering the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

we conclude that

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3},$$

i.e., $(x,y,z) = \frac{1}{\sqrt{3}}(a,b,c)$. The required maximum volume is therefore $8abc/3\sqrt{3}$.

Note that we can also, if we wish, deduce that $\lambda = 4abc/\sqrt{3}$, but this is of no practical relevance.