Summary of Results from Chapter 2: Poisson’s Equation

Physical Origins of Poisson’s Equation
Steady-state heat equation \( \nabla^2 T = -S(x)/k \)
Steady-state diffusion equation \( \nabla^2 \Phi = -S(x)/k \)
Electrostatic potential \( \nabla^2 \Phi = -\rho(x)/\varepsilon_0 \)
Gravitational potential \( \nabla^2 \Phi = 4\pi G\rho(x) \)
Flux (in each case) \( -k\nabla\Phi \)

Laplace’s Equation in 2D Plane Polars
\[
\nabla^2 \Phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0
\]
has general solution
\[
\Phi = A_0 + B_0 \theta + C_0 \ln \rho + \sum_{n=1}^{\infty} \left( A_n r^n + C_n r^{-n} \right) \cos n\theta + \sum_{n=1}^{\infty} \left( B_n r^n + D_n r^{-n} \right) \sin n\theta.
\]

Laplace’s Equation in 3D Spherical Polars, Axisymmetric Case
\[
\nabla^2 \Phi = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 0
\]
has general solution
\[
\Phi = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos \theta)
\]
where \( P_n \) are the Legendre polynomials. In particular, \( P_0(\cos \theta) = 1 \), \( P_1(\cos \theta) = \cos \theta \) and \( P_2(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1) \).

Uniqueness Theorem for Poisson’s Equation
If the problem \( \nabla^2 \Phi = \sigma(x) \), in a volume \( V \) with Dirichlet boundary conditions on the surface \( S \), has a solution for \( \Phi \), then that solution is unique.

Green’s Function
For a problem with Dirichlet boundary conditions, Green’s function \( G(x; x_0) \) satisfies
\[
\nabla^2 G = \delta(x - x_0) \quad \text{in } V,
\]
\[
G = 0 \quad \text{on } S.
\]
**Fundamental Solutions**

Green’s function when $V$ is all of space is

$$
\begin{cases}
\frac{1}{2\pi} \ln |x - x_0| + \text{constant} & \text{in 2D,} \\
-\frac{1}{4\pi|x - x_0|} & \text{in 3D.}
\end{cases}
$$

**Images in Circles and Spheres**

For Dirichlet boundary conditions, the image point is at

$$x_1 = \frac{a^2}{|x_0|^2} x_0,$$

with strength $-1$ in 2D and $-a/|x_0|$ in 3D.

**Green’s Identity**

$$\iint_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) dV = \iint_S (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot n dS$$

**The Integral Solution of Poisson’s Equation**

The solution to Poisson’s equation with Dirichlet boundary conditions,

$$\nabla^2 \Phi = \sigma \quad \text{in } V, \\
\Phi = f \quad \text{on } S,$$

is

$$\Phi(x_0) = \iiint_V \sigma(x) G(x; x_0) dV + \iint_S f(x) \frac{\partial G}{\partial n} dS$$

where $G(x; x_0)$ is Green’s function for the problem.

**Finite Differences**

A first order forward finite difference for $f'(x)$ is

$$f(x + \delta x) - f(x)$$

and a second order central finite difference for $f''(x)$ is

$$\frac{f(x + \delta x) - 2f(x) + f(x - \delta x)}{\delta x^2}.$$ 

**Discretization of Poisson’s Equation**

To solve $\nabla^2 \Phi = \sigma(x)$ in a rectangular domain $a \leq x \leq b, c \leq y \leq d$, introduce a grid with $x_i = a + i\delta x$, $y_j = c + j\delta y$. Denote the approximation to $\Phi(x_i, y_j)$ by $\Phi_{i,j}$; if $\delta x$ and $\delta y$ are equal then at an interior grid point $(i, j)$ a discretized version of Poisson’s equation is

$$\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j} = \sigma(x_i, y_j) \delta x^2.$$ 

*Mathematical Methods II*

*Natural Sciences Tripos Part IB*