Worked Example
Calculating Residues

Example: e^z/z^3

By expanding e^z as a Taylor series, we see that $f(z) = e^z/z^3$ has a Laurent expansion about $z = 0$ given by

$$z^{-3} + z^{-2} + \tfrac{1}{2}z^{-1} + \tfrac{1}{3!} + \cdots.$$

Hence the residue is $\frac{1}{2}$ (the coefficient of z^{-1}).

Alternatively, we note that f has a pole of order 3 at $z = 0$, so we can use the general formula for the residue at a pole:

$$\operatorname*{res}_{z=0} f(z) = \lim_{z\to 0}\left\{\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\big(z^3 f(z)\big)\right\} = \tfrac{1}{2}\lim_{z\to 0}\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}e^z\right\} = \tfrac{1}{2}.$$

Example: $e^z/(z^2-1)$

We have already calculated the Laurent expansion of $g(z) = e^z/(z^2-1)$ at $z = 1$:

$$\frac{e^z}{z^2-1} = \frac{e}{2}\left(\frac{1}{z-1} + \frac{1}{2} + \cdots\right),$$

so the residue is $e/2$.

Alternatively, we use the formula for the residue at a simple pole:

$$\operatorname*{res}_{z=1} g(z) = \lim_{z\to 1}\frac{(z-1)e^z}{z^2-1} = \lim_{z\to 1}\frac{e^z}{z+1} = e/2.$$

Example: $1/(z^8 - w^8)$

For any complex constant w, $h(z) = (z^8 - w^8)^{-1}$ has 8 simple poles, at $z = we^{n\pi i/4}$ $(n = 0, 1, \dots, 7)$. The residue at $z = w$, say, could be evaluated by factorizing $z^8 - w^8$ into its eight linear factors, but is most easily calculated using L'Hôpital's Rule:

$$\operatorname*{res}_{z=w} h(z) = \lim_{z\to w}\frac{z-w}{z^8-w^8} = \lim_{z\to w}\frac{1}{8z^7} = 1/8w^7.$$

Example: $1/\sinh \pi z$

$1/\sinh\pi z$ has a simple pole at $z = ni$ for all integers n (because the zeros of $\sinh z$ are at $n\pi i$ and are simple). We could use the Taylor series for $\sinh \pi z$, or the general residue formula: again using L'Hôpital's Rule, the residue is

$$\lim_{z\to ni}\frac{z-ni}{\sinh\pi z} = \lim_{z\to ni}\frac{1}{\pi\cosh\pi z} = \frac{1}{\pi\cosh n\pi i} = \frac{1}{\pi\cos n\pi} = (-1)^n/\pi.$$

Example: $1/\sinh^3 z$

We have seen that $\sinh^3 z$ has a zero of order 3 at $z = \pi i$, with Taylor series

$$\sinh^3 z = -(z-\pi i)^3 - \tfrac{1}{2}(z-\pi i)^5 + \cdots.$$

Therefore

$$\begin{aligned} 1/\sinh^3 z &= -(z-\pi i)^{-3}(1 + \tfrac{1}{2}(z-\pi i)^2 + \cdots)^{-1} \\ &= -(z-\pi i)^{-3}(1 - \tfrac{1}{2}(z-\pi i)^2 + \cdots) \\ &= -(z-\pi i)^{-3} + \tfrac{1}{2}(z-\pi i)^{-1} + \cdots. \end{aligned}$$

The residue is therefore $\frac{1}{2}$.