Worked Example
Geodesics on the Surface of a Sphere

Recall that in orthogonal curvilinear coordinates \((q_1, q_2, q_3)\),

\[
dr = h_1 \, dq_1 \, e_1 + h_2 \, dq_2 \, e_2 + h_3 \, dq_3 \, e_3.
\]

In spherical polar coordinates,

\[
dr = dr \, e_r + r \, d\theta \, e_\theta + r \sin \theta \, d\phi \, e_\phi.
\]

Without loss of generality, we may take the sphere to be of unit radius: the length of a path from \(A\) to \(B\) is then

\[
L = \int_A^B |dr| = \int_A^B \sqrt{d\theta^2 + \sin^2 \theta \, d\phi^2}
\]

where the path is described by the function \(\phi(\theta)\). Using Euler’s equation,

\[
\frac{d}{d\theta} \left( \frac{\partial}{\partial \phi'} \sqrt{1 + \sin^2 \theta \, \phi'^2} \right) = \frac{\partial}{\partial \phi} \sqrt{1 + \sin^2 \theta \, \phi'^2} = 0
\]

so that

\[
\frac{\sin^2 \theta \, \phi'}{\sqrt{1 + \sin^2 \theta \, \phi'^2}}
\]

is a constant, \(c\) say. Hence

\[
\phi' = \frac{c}{\sin \theta \sqrt{\sin^2 \theta - c^2}}
\]

and the problem reduces to integrating this with respect to \(\theta\).

Substitute \(u = \cot \theta\) so that \(du = -\cosec^2 \theta \, d\theta\). Then

\[
\phi = \int -\frac{c \, du}{\sqrt{1 - c^2 \cosec^2 \theta}}
\]

\[
= \int -\frac{c \, du}{\sqrt{1 - c^2(1 + u^2)}}
\]

\[
= \int -\frac{du}{\sqrt{a^2 - u^2}}
\]

where \(a = \frac{\sqrt{1 - c^2}}{c}\)

\[
= \cos^{-1}(u/a) + \phi_0
\]

where \(\phi_0\) is a constant of integration. Hence the geodesic path is given by

\[
cot \theta = a \cos(\phi - \phi_0)
\]

and the arbitrary constants \(a\) and \(\phi_0\) must be found using the end-points. This is a great circle path.