Worked Example

Contour Integration: Integration Round a Branch Cut

We wish to evaluate

\[ I = \int_{0}^{\infty} \frac{x^\alpha}{1 + \sqrt{2}x + x^2} \, dx \]

where \(-1 < \alpha < 1\) so that the integral converges. We will need a branch cut for \(z^\alpha\); we take this along the positive real axis and define

\[ z^\alpha = r^\alpha e^{i\alpha \theta} \]

where \(z = re^{i\theta}\) and \(0 \leq \theta < 2\pi\).

Consider

\[ \oint_{C} \frac{z^\alpha}{1 + \sqrt{2}z + z^2} \, dz \]

where the keyhole contour \(C\) consists of a large circle \(C_R\) of radius \(R\), a small circle \(C_\varepsilon\) of radius \(\varepsilon\) (to avoid the singularity of \(z^\alpha\) at \(z = 0\)) and two lines just above and below the branch cut, as shown.

The contribution from \(C_R\) is \(O(R^{\alpha-2}) \times 2\pi R = O(R^{\alpha-1}) \to 0\) as \(R \to \infty\).

The contribution from \(C_\varepsilon\) is (substituting \(z = \varepsilon e^{i\theta}\) on \(C_\varepsilon\))

\[ \int_{2\pi}^{0} \frac{\varepsilon^\alpha e^{i\alpha \theta}}{1 + \sqrt{2}\varepsilon e^{i\theta} + \varepsilon^2 e^{2i\theta}} i\varepsilon e^{i\theta} \, d\theta = O(\varepsilon^{\alpha+1}) \to 0 \]

as \(\varepsilon \to 0\).

The contribution from just above the branch cut is

\[ \int_{\varepsilon}^{R} \frac{x^\alpha}{1 + \sqrt{2}x + x^2} \, dx \to I \]

as \(\varepsilon \to 0\) and \(R \to \infty\). The contribution from just below the branch cut is

\[ \int_{R}^{\varepsilon} \frac{x^\alpha e^{2\alpha \pi i}}{1 + \sqrt{2}x + x^2} \, dx \to -e^{2\alpha \pi i} I \]

as \(\varepsilon \to 0\) and \(R \to \infty\).

Hence

\[ \oint_{C} \frac{z^\alpha}{1 + \sqrt{2}z + z^2} \, dz \to (1 - e^{2\alpha \pi i}) I \]

as \(\varepsilon \to 0\) and \(R \to \infty\).

But the integrand is equal to

\[ \frac{z^\alpha}{(z - e^{3\pi i/4})(z - e^{5\pi i/4})} \]
(by finding the roots of the quadratic), so the poles inside $C$ are at $e^{3\pi i/4}$ with residue $e^{3\alpha\pi i/4}/(\sqrt{2}i)$ and at $e^{5\pi i/4}$ with residue $e^{5\alpha\pi i/4}/(-\sqrt{2}i)$. Hence, taking the limits $\varepsilon \to 0$ and $R \to \infty$,

$$(1 - e^{2\alpha\pi i})I = 2\pi i \left( \frac{e^{3\alpha\pi i/4}}{\sqrt{2}i} + \frac{e^{5\alpha\pi i/4}}{-\sqrt{2}i} \right),$$

i.e.,

$$e^{\alpha\pi i}(e^{-\alpha\pi i} - e^{\alpha\pi i})I = \sqrt{2} \pi e^{\alpha\pi i}(e^{-\alpha\pi i/4} - e^{\alpha\pi i/4}).$$

We conclude that

$$I = \sqrt{2} \pi \frac{\sin(\alpha\pi/4)}{\sin(\alpha\pi)}.$$