Worked Example
Solving Differential Equations using the Laplace Transform and its Inverse

We shall solve \( \ddot{x} + x = 2 \sin t \)
for \( x(t) \), with initial conditions \( x(0) = 0, \dot{x}(0) = 2 \). Taking the Laplace transform with respect to time,

\[
(p^2 \ddot{x}(p) - px(0) - \dot{x}(0)) + \ddot{x}(p) = \frac{2}{p^2 + 1}.
\]

Using the initial conditions, we obtain

\[
p^2 \ddot{x} - 2 + \ddot{x} = \frac{2}{p^2 + 1}
\]
from which we deduce that

\[
\ddot{x} = \frac{2p^2 + 4}{(p^2 + 1)^2}.
\]

To invert this we write down the Bromwich inversion formula

\[
x(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{2p^2 + 4}{(p^2 + 1)^2} e^{pt} \, dp.
\]

The integrand has poles of order two at \( p = \pm i \), so we must have \( \gamma > 0 \) in order that the integration contour lies to the right of the singularities.

What are the residues at the poles? At \( p = i \), the residue is

\[
\lim_{p \to i} \frac{d}{dp} \left( \frac{2p^2 + 4}{(p + i)^2} e^{pt} \right) = \lim_{p \to i} \left( \frac{(p + i)(4p + (2p^2 + 4)t) - 2(2p^2 + 4)(p + i)}{(p + i)^3} e^{pt} \right)
= -\frac{1}{2}(t + 3i)e^{it}.
\]

Similarly, at \( p = -i \) the residue is \(-\frac{1}{2}(t - 3i)e^{-it}\).

As \( |p| \to \infty \), \( \ddot{x}(p) = O(|p|^{-2}) \to 0 \); hence for \( t > 0 \) we close the integration contour to the left, picking up the residues from the poles to obtain

\[
x(t) = -\frac{1}{2}(t + 3i)e^{it} - \frac{1}{2}(t - 3i)e^{-it}
= -\frac{1}{2}(2t \cos t + 3i(2i \sin t))
= 3 \sin t - t \cos t.
\]