Consider a uniform chain of length $L$, with mass per unit length $\rho$, hanging under gravity between the points $(-1, 1)$ and $(1, 1)$. It adopts a form of minimum potential energy, that is it minimises

$$\int_{-1}^{1} \rho g y \, dl \propto \int_{-1}^{1} y \sqrt{1 + y'^2} \, dx$$

subject to the prescribed length,

$$L = G[y] \equiv \int_{-1}^{1} \sqrt{1 + y'^2} \, dx.$$

This is equivalent to minimising $F - \lambda G$, i.e., to solving

$$\delta \int_{-1}^{1} (y - \lambda)\sqrt{1 + y'^2} \, dx = 0.$$

The integrand has no explicit $x$-dependence, so we use the first integral

$$c = (y - \lambda)\sqrt{1 + y'^2} - y'(y - \lambda) \frac{y'}{\sqrt{1 + y'^2}} = \frac{y - \lambda}{\sqrt{1 + y'^2}},$$

where $c$ is a constant, whence

$$x = \int \frac{c \, dy}{\sqrt{(y - \lambda)^2 - c^2}}.$$

Making the substitution $y = \lambda + c \cosh \theta$ we obtain

$$x = c \cosh^{-1} \left( \frac{y - \lambda}{c} \right) + x_0$$

where $x_0$ is an arbitrary constant of integration. Hence the solution is

$$y = \lambda + c \cosh \left( \frac{x - x_0}{c} \right),$$

which is a catenary.

We have three unknown constants, to be found using the equation for $y$ at each of the two end-points, together with the constraint equation. We immediately obtain $x_0 = 0$ by symmetry (or by solving the end-point equations for $x_0$). Now $y' = \sinh(x/c)$ and hence $\sqrt{1 + y'^2} = \cosh(x/c)$; so

$$L = \int_{-1}^{1} \cosh \frac{x}{c} \, dx = 2c \sinh \frac{1}{c}.$$
This equation must, in general, be solved numerically for \( c \) given \( L \); then \( \lambda \) can be found using the end-point at \((1, 1)\),

\[
1 = \lambda + c \cosh \frac{1}{c}.
\]

This completes the solution.