Worked Example
The Rayleigh–Ritz Method

The oscillations of a drum (e.g., a timpani, or more generally any circular membrane under tension and fixed at its boundary) obey Bessel's equation of order zero,

$$y'' + \frac{1}{x}y' + \lambda y = 0,$$

in $0 \leqslant x \leqslant 1$, with boundary conditions that y should be non-singular at $x = 0$ and that $y(1) = 0$. Here $\lambda = \omega^2/c^2$ where ω is the frequency of oscillation of the drum and c is the wave speed. (This equation may be derived by converting the two-dimensional wave equation on the surface of the drum into plane polar coordinates and assuming a radially symmetric solution with a fixed frequency ω of oscillation.)

A drum can oscillate at many different frequencies, corresponding to the different eigenvalues of this Sturm–Liouville problem; but the *fundamental* (i.e., lowest) frequency is of the greatest interest since this is the one a listener will hear. (When a drum is struck, *all* of the possible frequencies are produced to varying extents, but the *harmonics*, i.e., the higher frequencies, usually decay rapidly leaving only the fundamental.) It is therefore natural to use the Rayleigh–Ritz method to estimate the lowest eigenvalue of Bessel's equation (and thereby estimate the fundamental frequency).

Before we can proceed, we **must** put the equation into standard Sturm–Liouville self-adjoint form. By inspection we see that the appropriate equation is

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \lambda xy.$$

The equivalent variational problem is therefore that $F[y] = \int_0^1 xy'^2\,\mathrm{d}x$ is stationary subject to $G[y] = \int_0^1 xy^2\,\mathrm{d}x = 1$. We shall use a trial solution of the form

$$y_{\text{trial}} = a + bx^2 + cx^4$$

(chosen because we anticipate that the lowest eigenvalue corresponds to a solution which is *even* in x). This trial solution trivially satisfies the boundary condition at $x = 0$, and satisfies the condition at $x = 1$ so long as $a + b + c = 0$.

We now calculate

$$\begin{aligned} F[y_{\text{trial}}] = \int_0^1 xy_{\text{trial}}'^2\,\mathrm{d}x &= \int_0^1 x(2bx + 4cx^3)^2\,\mathrm{d}x \\ &= b^2 + \tfrac{8}{3}bc + 2c^2 \end{aligned}$$

and

$$\begin{aligned} G[y_{\text{trial}}] = \int_0^1 xy_{\text{trial}}^2\,\mathrm{d}x &= \int_0^1 x(a + bx^2 + cx^4)^2\,\mathrm{d}x \\ &= \tfrac{1}{2}a^2 + \tfrac{1}{2}ab + \tfrac{1}{6}(b^2 + 2ac) + \tfrac{1}{4}bc + \tfrac{1}{10}c^2 \\ &= \tfrac{1}{6}b^2 + \tfrac{5}{12}bc + \tfrac{4}{15}c^2, \end{aligned}$$

using $a = -b - c$.

We must *either* minimise F/G — which turns out to be rather messy algebraically — *or* minimise F subject to $G = 1$. We choose the latter; hence we minimise $F - \lambda G$ with respect to both b and c. So

$$0 = \frac{\partial}{\partial b}(F - \lambda G) = (2 - \tfrac{1}{3}\lambda)b + (\tfrac{8}{3} - \tfrac{5}{12}\lambda)c \tag{1}$$

and

$$0 = \frac{\partial}{\partial c}(F - \lambda G) = (\tfrac{8}{3} - \tfrac{5}{12}\lambda)b + (4 - \tfrac{8}{15}\lambda)c. \tag{2}$$

Eliminating b and c from these equations, and rearranging, we find that

$$3\lambda^2 - 128\lambda + 640 = 0,$$

which has two solutions

$$\lambda = \tfrac{1}{3}\big(64 \pm \sqrt{2176}\,\big) = 5.7841\ldots \text{ or } 36.8825\ldots.$$

We recall that the eigenvalues of the Sturm–Liouville equation are given by the values of the Lagrange multiplier λ. Therefore the lowest eigenvalue of this problem is approximately 5.7841 (and certainly no larger). We could find the corresponding values of b and c (and hence a) by substituting this value of λ into either equation (1) or (2) (both give the same result); note that we find only the ratio $a : b : c$ because the normalisation of y_{trial} is not important.

In fact, the true value of the lowest eigenvalue is $5.7832\ldots$, so the Rayleigh–Ritz method has produced an extremely good estimate.