Summary of Results from Chapter 1: Variational Methods

Stationary Values in Finite Dimensions

The stationary points of a function $f(\mathbf{x})$ occur where $\nabla f = \mathbf{0}$. The stationary points of $f(\mathbf{x})$ subject to a constraint $g(\mathbf{x}) = c$, where c is a constant, can be found by solving $\nabla(f - \lambda g) = \mathbf{0}$ and eliminating λ using the constraint.

Euler's Equation

The function $y(x)$ which makes the functional

$$\int_a^b f(x, y, y')\,\mathrm{d}x$$

stationary, with fixed values of $y(a)$ and $y(b)$, satisfies Euler's equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}.$$

First Integral

If $f(x, y, y')$ has no explicit x-dependence, then $f - y'\dfrac{\partial f}{\partial y'}$ is a constant of the motion.

Fermat's Principle

In the geometric optics approximation, light rays passing through a medium of variable refractive index $\mu(\mathbf{r})$ take that path which minimises the optical path length

$$\int \mu(\mathbf{r})\,\mathrm{d}l.$$

Hamilton's Principle of Least Action

The Lagrangian of a mechanical system is defined by $\mathscr{L} = T - V$ where T is the kinetic energy and V is the potential energy. The motion of the system is such as to minimise the action

$$\mathscr{S} = \int \mathscr{L}\,\mathrm{d}t.$$

If the configuration of the system is described by n generalised coordinates $q_1, \ldots, q_n$ then the system obeys the Euler–Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathscr{L}}{\partial q_i},$$

for $i = 1, \ldots, n$.

If $\mathscr{L}$ has no explicit time-dependence then

$$\mathscr{L} - \sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathscr{L}}{\partial \dot{q}_i}$$

is a constant of the motion. If T is a homogeneous quadratic in the $\dot{q}_i$, V does not depend on the $\dot{q}_i$, and both T and V have no explicit time-dependence, then this first integral is proportional to the total energy $T + V$, which is therefore conserved.

Calculus of Variations with Constraint

To find the stationary values of a functional $F[y]$ subject to a constraint $G[y] = c$, where c is a constant, instead find the stationary values of $F - \lambda G$ without constraint and then eliminate λ using $G = c$.

Variational Principle for Sturm–Liouville Equations

The Sturm–Liouville problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}(py') + qy = \lambda wy$$

in $a < x < b$ is equivalent to the variational problem of finding the stationary values of the functional

$$F[y] = \int_a^b (py'^2 + qy^2)\,\mathrm{d}x$$

subject to $G[y] = 1$ where

$$G[y] = \int_a^b wy^2\,\mathrm{d}x,$$

and the eigenvalues of the Sturm–Liouville problem are then given by the corresponding stationary values of F. The eigenvalues are also equal to the values of the Lagrange multiplier used in the constrained variation.

This Sturm–Liouville problem is also equivalent to the variational problem of finding the stationary values of the Rayleigh quotient

$$\Lambda[y] = \frac{F[y]}{G[y]},$$

and the corresponding stationary values of Λ are the eigenvalues.

The Rayleigh–Ritz Method

To estimate the lowest eigenvalue λ_0 of a Sturm–Liouville problem, use a trial function y_{trial} with some number of adjustable parameters. Calculate $\Lambda[y_{\text{trial}}]$ and minimise this with respect to the parameters. The result will be an approximation to λ_0, and will be greater than or equal to it.