Worked Example

Steady-State Temperature Distribution in a Cylinder

An infinitely long cylinder of radius \( a \) is heated on its boundary as shown. The steady-state temperature \( T(r, \theta) \) (note no dependence on \( z \)) satisfies

\[
\nabla^2 T = 0 \quad \text{in} \ r < a
\]

subject to

\[
T(a, \theta) = \begin{cases} 
+T_0 & 0 \leq \theta < \pi, \\
-T_0 & \pi \leq \theta < 2\pi. 
\end{cases}
\]

The general solution for plane polar coordinates applies; we choose to use it in its second form as given in the lecture notes. We require that the temperature be finite at \( r = 0 \) for a physically realistic solution: so \( C_0 = 0 \), and also, for all negative \( n \), \( A_n = B_n = 0 \) (since they are the coefficients of \( r^n \{ \cos \} n\theta \)). Finally, \( T \) must be periodic in \( \theta \) (i.e., not multi-valued), so \( B_0 = 0 \). Hence

\[
T(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).
\]

On \( r = a \) this gives

\[
T(a, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n a^n \cos n\theta + B_n a^n \sin n\theta).
\]

This is a standard Fourier series, so we may calculate the Fourier coefficients using the standard formulae:

\[
A_0 = \frac{1}{2\pi} \int_0^{2\pi} T(a, \theta) \, d\theta = 0 \quad \text{(by anti-symmetry of } T(a, \theta))
\]

\[
A_n a^n = \frac{1}{\pi} \int_0^{2\pi} T(a, \theta) \cos n\theta \, d\theta = 0
\]

\[
B_n a^n = \frac{1}{\pi} \int_0^{2\pi} T(a, \theta) \sin n\theta \, d\theta
\]

\[
= \frac{1}{\pi} \int_0^{\pi} T_0 \sin n\theta \, d\theta - \frac{1}{\pi} \int_{\pi}^{2\pi} T_0 \sin n\theta \, d\theta
\]

\[
= \begin{cases} 
4T_0/n\pi & n \text{ odd,} \\
0 & n \text{ even.}
\end{cases}
\]

Hence the final solution for all \( r \) and \( \theta \) is

\[
T = \frac{4T_0}{\pi} \sum_{n \text{ odd}} \frac{r^n}{na^n} \sin n\theta.
\]