Worked Example

Heat Source near an Insulated Wall

Hold a heat source of strength \( Q \) at \( x_0 = (x_0, y_0, z_0) \) near an insulated plane wall, i.e., one through which no heat can pass, at \( x = 0 \). We must then have no component of heat flux through the wall; i.e., \( \mathbf{n} \cdot (-k \nabla T) = 0 \) on the wall. Therefore we must solve

\[
\nabla^2 T = -\frac{Q}{k} \delta(x - x_0) \quad \text{in } x > 0
\]

subject to

\[
\frac{\partial T}{\partial n} = 0 \quad \text{on } x = 0.
\]

This is a problem with Neumann (rather than Dirichlet) boundary conditions.

We use the method of images. Introduce an image source of strength \(+Q\) at \( x_1 = (-x_0, y_0, z_0)\). (Note that for Dirichlet boundary conditions we would have used \(-Q\) for the strength of the image.) Because \( \nabla T \) is radial from each source, the total \( \nabla T \) (from the two sources combined) must have zero component perpendicular to the wall. Hence we have \( \partial T/\partial n = 0 \) as required. Therefore (by uniqueness) the solution is

\[
T = \frac{Q}{4\pi k} \left\{ \frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{|\mathbf{x} - \mathbf{x}_1|} \right\}.
\]