Example Sheet 1: Variational Methods

1 A coal box, in the shape of a cuboid, is to be placed flush against an outside wall, so that only its top, front and two sides are visible. The owner wishes the box to contain at least a certain volume $V$ of coal, but also wishes to minimise the visible surface area in order to avoid the box becoming an eyesore. What lengths should be chosen for the sides?

2 The temperature within and on the surface of a sphere of unit radius is given by $T(x, y, z) = x(y + z)$. Find the minimum and maximum temperature.

3 Find the geodesics on a cylinder of radius $a$.

4 The new Mayor of London is devoted to schemes for energy saving and wishes to design a fuel-less tube transport system driven by gravity. He proposes that vehicles should travel in frictionless underground tunnels, being dropped from rest at their point of departure $A$ (Waterloo), and then allowed to run freely until they arrive at their destination $B$ (Paddington), a distance $L$ apart at the same level. Show that, neglecting variations in gravity, the minimum travel time is $\sqrt{2\pi L/g}$, and that the tunnels at the end-points should be vertical. (Which should make the tube journey that bit more exciting.)

5 State Fermat’s principle governing the paths traced by light rays and explain the conditions under which it applies. Given that in a horizontally stratified medium the refractive index is given by $\mu(z) = \sqrt{a - b z}$, where $z$ is the height and $a$ and $b$ are positive constants, prove that light rays travelling in a vertical plane follow inverted parabolas. Show further that all such parabolas have their directrix in the plane $z = a/b$. [The directrix of a parabola in standard form, $y^2 = 4ax$, is the line $x = -a$.]

6 A particle of unit mass moves in a plane with polar coordinates $(r, \theta)$, under the influence of a central force derived from a potential $V(r)$. Write down the action functional for this problem and use Hamilton’s principle to find differential equations for $r(t)$ and $\theta(t)$. Give a physical interpretation of these equations. Given that the particle’s trajectory is $r = a \sin \theta$ for some constant $a$, deduce that (up to an arbitrary additive constant) $V \propto r^{-4}$.

7 If $\mu(r) = |\nabla f(r)|$ for some function $f$, show that $\int_A^B \mu \, dl$ between two points $A$ and $B$ is at least $f(B) - f(A)$, with equality if and only if the path of integration lies orthogonal to the family of surfaces $f = \text{constant}$. Deduce that such orthogonal trajectories satisfy Fermat’s principle.
8 A soap film is bounded by two circular wires at \( r = a, z = \pm b \) in cylindrical polar coordinates \((r, \theta, z)\). Assuming that the soap surface is cylindrically symmetric, show that the equation of the surface of minimal area is

\[
\frac{r}{c} = \cosh \left( \frac{z}{c} \right)
\]

where \( c \) satisfies the condition \( a/c = \cosh(b/c) \). Show graphically that this condition has no solution for \( c \) if \( b/a \) is larger than a certain critical ratio. What happens to the soap surface as \( b/a \) is increased from below this ratio to above it?

9 An area is enclosed by joining two fixed points a distance \( a \) apart on a straight wall with a given length \( l \) of flexible fencing \((a < l < \pi a)\). How is the area maximised?

10 Show from first principles that the equivalent of Euler’s equation for the function \( x(t) \) which extremises the integral

\[
\int_{t_1}^{t_2} f(t, x, \dot{x}, \ddot{x}) \, dt
\]

with fixed values of both \( x(t) \) and \( \dot{x}(t) \) at \( t = t_1 \) and \( t_2 \) is

\[
\frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial f}{\partial \ddot{x}} \right) = 0.
\]

Hence find the function \( x(t) \) with \( x(1) = 1, \dot{x}(1) = -2, x(2) = \frac{1}{4} \) and \( \dot{x}(2) = -\frac{1}{4} \) that minimises \( \int_1^2 t^4 \{\ddot{x}(t)\}^2 \, dt \).

11 Consider the Sturm–Liouville problem

\[-(1 + x^2)y'' - 2xy' = \lambda y \]

with \( y(\pm 1) = 0 \). Use the Rayleigh–Ritz method to obtain an upper bound on the lowest eigenvalue by using the trial function \( y_1 = 1 - x^2 \). Show that a better bound is obtained from the trial function \( y_2 = \cos(\pi x/2) \) and explain how a further improvement could be achieved by considering \( y_1 \) and \( y_2 \) in combination. [\( \int_{-1}^1 x^2 \sin^2(\pi x/2) \, dx = \frac{1}{3} + \frac{2}{\pi^2} \].]

12 The differential equation governing small transverse displacements \( y(x) \) of a string with fixed end-points at \( x = 0 \) and \( x = \pi \) is

\[ y'' + \omega^2 f(x) y = 0 \]

where \( \omega \) is the angular frequency of the vibration and \( f \) is a positive function. Show that the allowed values of \( \omega^2 \) are given by the stationary values of

\[ \frac{\int_0^\pi y'^2 \, dx}{\int_0^\pi f(x) y^2 \, dx} \]

Use this fact to find an approximate value for the angular frequency of the fundamental mode when \( f(x) = 1 + \sin x \).
Show that $\psi_0 = \exp\left(-\frac{1}{2}x^2\right)$ is an eigenfunction of the operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + (x^2 - 1)$$

acting on functions $\psi(x)$ for which $\psi \to 0$ as $|x| \to \infty$, and find the corresponding eigenvalue $\lambda_0$. This is in fact the lowest eigenvalue of the problem.

Use the Rayleigh–Ritz method with trial function

$$\tilde{\psi}_0 = \begin{cases} b(a^2 - x^2) & |x| < a \\ 0 & |x| \geq a \end{cases}$$

where $a$ and $b$ are adjustable constants, to obtain the approximation

$$\tilde{\lambda}_0 = \sqrt{\frac{10}{7}} - 1$$

to $\lambda_0$. Comment on the sign of $\tilde{\lambda}_0 - \lambda_0$.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at reh10@damtp.cam.ac.uk.