Pigeonhole I11

Mathematical Methods II — Dr R. E. Hunt
Natural Sciences Tripos Part IB — Lent 2002

Example Sheet 2: Poisson's Equation

1 Using the method of separation of variables in Cartesian coordinates, solve the following for $\Phi(x,y)$:

$$\begin{aligned} \nabla^2\Phi &= 0 && \text{in } 0 < x < 1 \text{ and } y > 0, \\ \Phi &= 0 && \text{on } x = 0 \text{ or } x = 1, \text{ for } y > 0, \\ \Phi &= x(1-x) && \text{on } y = 0, \\ \Phi &\to 0 && \text{as } y \to \infty. \end{aligned}$$

Hence find

$$\int_0^1 \left.\frac{\partial\Phi}{\partial y}\right|_{y=0} \mathrm{d}x,$$

leaving your answer as an infinite sum.

2 Solve the following for $\Phi(r,\theta)$ inside a sphere of radius a:

$$\begin{aligned} \nabla^2\Phi &= 0 \quad \text{in } r < a, \\ \Phi &= 1 + \cos\theta + \cos^2\theta \quad \text{on } r = a. \end{aligned}$$

Include in your answer a full derivation of any general result you may use.

3 A solute of diffusivity k both surrounds and fills the interior of a cylinder of radius a. The concentration of the solute at a point (r,θ) in plane polar coordinates is denoted by $\Phi(r,\theta)$. The boundary of the cylinder is permeable, and solute diffuses through it in such a way that at the point (a,θ) the normal component of the flux of solute is given by $F\cos 2\theta$, where F is a constant. Find the steady-state distribution of solute within the interior of the cylinder.

Why does your answer include an arbitrary constant? How is it possible for there to be a constant flux of solute through the boundary and yet for the concentration within to remain fixed?

4 A single-valued potential Φ satisfies $\nabla^2\Phi = 0$ inside a cylindrical annulus $a < r < b$. On $r = b$, $\Phi = 0$ for all θ; and on $r = a$, $\Phi = 0$ for $-\pi < \theta \leqslant 0$ and $\Phi = 1$ for $0 < \theta \leqslant \pi$. Find $\Phi(r,\theta)$ inside the annulus.

5 Heat flows past a poorly conducting cylinder of radius a whose axis lies on the z-axis. Find the temperature distribution $T(r,\theta)$, which satisfies the following equations:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}=0 \quad \text{in } r>a \text{ and in } r<a,$$

$$T\sim Gr\cos\theta \quad \text{as } r\to\infty,$$

$$T(a^+,\theta)=T(a^-,\theta),$$

$$\frac{\partial T}{\partial r}(a^+,\theta)=\beta\,\frac{\partial T}{\partial r}(a^-,\theta),$$

where G and β $(0<\beta<1)$ are constants. Can you explain the origin of each of these equations? [*'a^+' here denotes the limit as $r\to a$ from above; 'a^-' from below.*]

6 Consider an electric charge density distribution

$$\rho=\begin{cases} r^{-1}\cos\theta & \text{in } 0<r<a\\ 0 & \text{in } r\geqslant a\end{cases}$$

where (r,θ,ϕ) are spherical polar coordinates. By writing the solution as a function of r multiplied by a *suitably chosen* function of θ, find the (axisymmetric) electrostatic potential both inside and outside the region $r<a$. (The potential and its radial derivative must be continuous everywhere; why?)

[*You may wish to start by proving that the general solution of the differential equation* $r^2R''+2rR'-2R=\alpha r$, *where* R *is a function of* r *and* α *is a constant, is* $R=Ar+Br^{-2}+(\alpha/3)\,r\ln r$.]

7 Show that for each of the following equations governing the scalar field $\Phi(\mathbf{x})$ in a volume V, subject to the given boundary conditions on S, the surface of V, there is a unique solution for Φ:

(i) $\nabla^2\Phi=0$ subject to $\partial\Phi/\partial n$ specified on S (Neumann boundary conditions), the solution only being unique up to a constant in this case;

(ii) the Klein–Gordon equation $\nabla^2\Phi-m^2\Phi=0$, where m is a non-zero real constant, subject to Dirichlet boundary conditions;

(iii) the Klein–Gordon equation subject to Neumann boundary conditions.

8 The two-dimensional quarter-plane $x>0$, $y>0$ is occupied by a single source of heat of strength Q positioned at the point (x_0,y_0). At $x=0$ there is a plane conducting wall which is held at a temperature T_0. At $y=0$ there is an insulated wall across which no heat can flow (i.e., the heat flux normal to the wall must vanish). Write down the equation and boundary conditions satisfied by the temperature field.

Use the method of images to find the temperature field. Hence show that the magnitude of the heat flux across the wall at the point $(0,y)$ is

$$\frac{Qx_0}{\pi}\left\{\frac{1}{x_0^2+(y-y_0)^2}+\frac{1}{x_0^2+(y+y_0)^2}\right\}.$$

How would you calculate the *total* heat radiated across the wall $x=0$?

9 Use the method of images in plane polar coordinates to find Green's function for the Laplacian operator (∇^2) with Dirichlet boundary conditions in the following two-dimensional domains. [*You should prove any results concerning image points given in lectures.*]

(i) The disc $0 \leqslant r < a$.

(ii) The half-disc $0 < r < a$, $0 < \theta < \pi$.

10 Define the Green's function $G(\mathbf{x}; \boldsymbol{\xi})$ for the three-dimensional Dirichlet problem

$$\nabla^2 u = 0 \quad \text{in } V, \qquad u = f \quad \text{on } S,$$

where S is the closed surface bounding the volume V. Show that

$$u(\boldsymbol{\xi}) = \iint_S f(\mathbf{x}) \frac{\partial G}{\partial n}(\mathbf{x}; \boldsymbol{\xi})\, \mathrm{d}S$$

where $\partial/\partial n$ denotes differentiation along the outward normal on S. What is the equivalent statement in two dimensions?

Show that a solution of $\nabla^2 u = \delta(\mathbf{x})$ in two dimensions ($r^2 = x^2 + y^2$) is $\frac{1}{2\pi} \ln r$. Hence derive the form of Green's function for Laplace's equation in the two-dimensional region $-\infty < x < \infty$, $y > 0$ with Dirichlet boundary conditions. Show how Green's function is used to obtain the solution u of Laplace's equation in this region when $u \to 0$ as $r \to \infty$, $u = 0$ on $y = 0$ for $|x| > 1$, and $u = 1$ on $y = 0$ for $|x| \leqslant 1$.

11 Using the integral expression for the solution of Poisson's equation, evaluate the gravitational potential $\Phi(r, z)$ on the symmetry axis $r = 0$ due to a thin disc of uniform density and total mass M lying in the plane $z = 0$ and occupying the region $r \leqslant a$, where (r, θ, z) are *cylindrical* polar coordinates. Find the expansion of your expression as a series for $|z| \gg a$ accurate to $O(z^{-5})$.

Now consider the general solution of Laplace's equation in *spherical* polar coordinates at large distances r_s from the origin. By identifying the terms in this solution with those in your expansion, obtain an expression for Φ which is valid off the symmetry axis. [*Recall that* $P_n(\cos\theta) = 1$ *when* $\cos\theta = 1$.]

12 State a version of Green's identity applicable to a plane surface S bounded by a closed curve C. Use this identity and the Green's function from question 9(i) to show that the solution of

$$\begin{aligned} \nabla^2 \Phi &= 0 && \text{in } 0 \leqslant r < a, \\ \Phi &= \Psi(\theta) && \text{on } r = a,\ 0 \leqslant \theta < 2\pi, \end{aligned}$$

is

$$\Phi(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\Psi(\theta')\, \mathrm{d}\theta'}{a^2 - 2ar\cos(\theta - \theta') + r^2}.$$

13 Use the solution of the previous question to show that in $0 \leqslant r < 1$,

$$\int_0^{2\pi} \frac{\mathrm{d}\theta'}{1 - 2r\cos(\theta - \theta') + r^2} = \frac{2\pi}{1 - r^2},$$

$$\int_0^{2\pi} \frac{\sin\theta' \, \mathrm{d}\theta'}{1 - 2r\cos(\theta - \theta') + r^2} = \frac{2\pi r \sin\theta}{1 - r^2},$$

$$\int_0^{2\pi} \frac{\cos^2\theta' \, \mathrm{d}\theta'}{1 - 2r\cos(\theta - \theta') + r^2} = \frac{2\pi r^2 \cos^2\theta}{1 - r^2} + \pi.$$

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@damtp.cam.ac.uk`.