Pigeonhole I12

Mathematical Methods II — Dr R. E. Hunt
Natural Sciences Tripos Part IB — Lent 2002

Hints and Solutions for Example Sheet 2: Poisson's Equation

1 Use methods from Part IA. You should obtain a series in which $\sin n\pi x$, but not $\cos n\pi x$, appears; you must therefore treat this as a Fourier *sine* series, for which the formulae are slightly different from those for a standard Fourier series. The solution is

$$\Phi(x,y) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3} \sin n\pi x \, e^{-n\pi y}.$$

Substituting this infinite sum into the integral gives

$$\int_0^1 \left.\frac{\partial \Phi}{\partial y}\right|_{y=0} \mathrm{d}x = -\frac{16}{\pi^3} \sum_{n \text{ odd}} \frac{1}{n^3}.$$

2 Use the general axisymmetric solution in spherical polar coordinates, and retain only relevant P_n. Note that Φ must be finite at the origin. The solution is $\Phi(r,\theta) = \frac{4}{3} + (r/a)\cos\theta + (r^2/a^2)(\cos^2\theta - \frac{1}{3})$.

3 Use the general solution in cylindrical polars in the region $r < a$, and note that Φ must be finite at the origin. The boundary condition is that $\mathbf{n}\,.\,(-k\nabla\Phi) = F\cos 2\theta$ on $r = a$; write this in a simpler form. On $r = a$, you can use the fact that Fourier series are unique to work out the constants A_n and B_n very quickly. The solution is

$$\Phi = A_0 - \frac{Fr^2\cos 2\theta}{2ka}.$$

To answer the last part of the question, you may like to consider the *total* flux passing through the boundary of the cylinder.

4 Use the general solution in its less concise form; this is a standard Fourier series (albeit with complicated coefficients of $\cos n\theta$ and $\sin n\theta$ which depend on r). On $r = b$, all the Fourier coefficients must vanish. On $r = a$, you can evaluate them by finding the Fourier series of the given boundary condition. This should enable you to solve for A_0, C_0, A_n, B_n, C_n and D_n simultaneously. The solution is

$$\Phi(r,\theta) = \frac{\ln(r/b)}{2\ln(a/b)} + \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{2}{n\pi} \frac{(b/r)^n - (r/b)^n}{(b/a)^n - (a/b)^n} \sin n\theta.$$

5 Use the general solution twice, with *different* coefficients in $r < a$ and $r > a$, and use the conditions as $r \to 0$ and $r \to \infty$ to eliminate some terms. Use the final two equations to relate the two infinite series at $r = a$; remember that you can equate coefficients of $\cos n\theta$, etc. (Why?) The solution is

$$T = \begin{cases} A_0 + 2G(1+\beta)^{-1} r\cos\theta & r < a, \\ A_0 + G\left(r + \dfrac{1-\beta}{1+\beta}\dfrac{a^2}{r}\right)\cos\theta & r > a. \end{cases}$$

The final equation given in the question results from considering heat flux at $r = a$.

6 Write down Poisson's equation $\nabla^2\Phi = -\rho/\epsilon_0$, and let $\Phi(r,\theta) = R(r)\Theta(\theta)$. In this case, you can pick a particular form for Θ so that each term in Poisson's equation contains exactly the same function of θ; hence Θ cancels throughout. This leaves you with the ordinary differential equation for $R(r)$ given in the hint, with $\alpha = -1/\epsilon_0$ in $r < a$ and zero in $r \geqslant a$. (The solution of this differential equation, as given in the hint, can be obtained using the method of "complementary function plus particular integral"; you may need to use the substitution $u = \ln r$ to obtain the particular integral.)

Note that there are different values of A and B in $r < a$ and $r \geqslant a$. In $r < a$, $B = 0$ in order that the potential is finite at the origin. In $r \geqslant a$, $A = 0$ in order that the potential should tend to zero as $r \to \infty$. Use continuity of R and R' at $r = a$ to find the values of the two remaining arbitrary constants.

The solution is

$$\Phi(r,\theta) = \begin{cases} \big(1 - 3\ln(r/a)\big) r\cos\theta/9\epsilon_0 & r < a \\ a^3\cos\theta/9\epsilon_0 r^2 & r \geqslant a \end{cases}$$

[*Two notes for advanced students.* Firstly, although ρ has a singularity at $r = 0$, the total charge in any given volume is in fact finite, because a volume integral in spherical polars must include the Jacobian $r^2\sin\theta$. Hence we do expect the potential at the origin to be finite. Secondly, as $r \to \infty$, you should think about what it would mean if you did not insist on the potential tending to zero: it would lead to the existence of a constant electric field far from the charges, which is quite possible, but which would not be caused by the charges themselves and should therefore be ignored unless specified otherwise in the question. Ask your supervisor if either of these notes doesn't make sense!]

7 The method required is, in each case, almost identical to that used in lectures to prove uniqueness for Poisson's equation with Dirichlet boundary conditions.

8 The equation is $\nabla^2 T = -(Q/k)\delta(\mathbf{x} - \mathbf{x}_0)$ in $x > 0$, $y > 0$, subject to $T = T_0$ on $x = 0$ and $\partial T/\partial n = 0$ on $y = 0$. You will need 3 image sources, one with positive strength and two with negative strength. The solution is

$$T = T_0 - \frac{Q}{2\pi k}\ln\frac{|\mathbf{x} - \mathbf{x}_0||\mathbf{x} - \mathbf{x}_1|}{|\mathbf{x} - \mathbf{x}_2||\mathbf{x} - \mathbf{x}_3|}$$

where $\mathbf{x}_1 = (x_0, -y_0)$, $\mathbf{x}_2 = (-x_0, y_0)$ and $\mathbf{x}_3 = (-x_0, -y_0)$. To find the heat flux, calculate $-k\,\partial T/\partial n$ on the y-axis. To calculate the total heat radiated, either integrate the given expression for the heat flux over the wall, or use the Divergence Theorem appropriately, to obtain the answer Q.

9 (i) Use the inverse point and the corresponding Green's function mentioned in lectures. To show that this function vanishes on $r = a$, use, for example, the identity $|\mathbf{x} - \mathbf{x}_1|^2 = |\mathbf{x}|^2 - 2\mathbf{x}\,.\,\mathbf{x}_1 + |\mathbf{x}_1|^2$. The solution, complete with the correct value of the constant which was not specified in lectures, is

$$G(\mathbf{x}; \mathbf{x}_0) = \frac{1}{2\pi} \ln \frac{a|\mathbf{x} - \mathbf{x}_0|}{|\mathbf{x}_0||\mathbf{x} - \mathbf{x}_1|}$$

where $\mathbf{x}_1 = (a^2/|\mathbf{x}_0|^2)\mathbf{x}_0$.

(ii) Using two extra image points we obtain

$$G(\mathbf{x}; \mathbf{x}_0) = \frac{1}{2\pi} \ln \frac{|\mathbf{x} - \mathbf{x}_0||\mathbf{x} - \mathbf{x}_1'|}{|\mathbf{x} - \mathbf{x}_0'||\mathbf{x} - \mathbf{x}_1|}$$

where $\mathbf{x}_0'$ and $\mathbf{x}_1'$ are the reflections of $\mathbf{x}_0$ and $\mathbf{x}_1$ in the x-axis.

10 The equivalent statement in 2D is exactly the same, but with V replaced by S and S by C.

The particular case given is similar to a worked example; the solution is

$$\begin{aligned} u(x_0, y_0) &= \frac{y_0}{\pi} \int_{-1}^{1} \frac{\mathrm{d}x}{(x - x_0)^2 + y_0^2} \\ &= \frac{1}{\pi} \left\{ \tan^{-1}\left(\frac{1 - x_0}{y_0}\right) + \tan^{-1}\left(\frac{1 + x_0}{y_0}\right) \right\}. \end{aligned}$$

11 The density is proportional to $\delta(z)$; use plane polars on the plane $z = 0$ to evaluate the "volume" integral which arises from the integral solution of Poisson's equation in all space. The solution in cylindrical polars is

$$\Phi(0, z) = -\frac{2GM}{a^2}(\sqrt{z^2 + a^2} - |z|).$$

The expansion for $|z| \gg a$ is then

$$-GM|z|\left(\frac{1}{z^2} - \frac{a^2}{4z^4} + \frac{a^4}{8z^6}\right) + O\left(\frac{a^6}{|z|^7}\right).$$

Compare with the general axisymmetric solution in spherical polars (r_s, θ_s, ϕ_s) on the axis (where $\theta_s = 0$); the coefficients must match. The solution valid off the axis is

$$\Phi(r_s, \theta_s) = GM\left(-\frac{1}{r_s} + \frac{a^2}{4r_s^3} P_2(\cos\theta_s) - \frac{a^4}{8r_s^5} P_4(\cos\theta_s)\right) + O\left(\frac{a^6}{r_s^7}\right).$$

12 The version of Green's identity required is simply the normal version with V replaced by S and S by C. Write $\mathbf{x}$, $\mathbf{x}_0$, $\mathbf{x}_1$ in terms of 2D polar coordinates, and calculate $|\mathbf{x} - \mathbf{x}_0|$, etc., in order to obtain an expression for G. Then derive the integral solution of Poisson's equation (in 2D) just as normal: you will need to calculate $\frac{\partial G}{\partial r}\Big|_{r=a}$.

13 For each part, choose a suitable function Ψ and work out what the solution for Φ must be by inspection (using uniqueness). Then apply the formula derived in the previous question.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@damtp.cam.ac.uk`.