Mathematical Tripos Part III

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Statistical Field Theory

Examples 2

Statistical Models and Landau Theory

(1) By considering $< \sigma_0 >$ in equilibrium show that in the mean field approximation to the Ising model in D-dimensions the equilibrium magnetisation is given by the solution to

$$M = \tanh \beta(qJM + h)$$

where $\beta = 1/kT$, $q = 2D$, $J(> 0)$ is the nearest neighbour coupling constant and h is the applied magnetic field.

The free energy in this approach is

$$A = -kT \log \left[2 \cosh \beta(qJM + h)\right] \; + \; \frac{1}{2}qJM^2.$$

Show that the expression for the equilibrium magnetization above can also be obtained by minimizing A with respect to M.

The critical exponent α governs the divergence in the specific heat:

$$C \;=\; T\left(\frac{\partial^2 A}{\partial T^2}\right)_{h=0}, \qquad C \sim |T - T_c|^{-\alpha} \quad T \to T_c. \tag{1}$$

Using the expression for M above show that the free energy in equilibrium for $h = 0$ is

$$A \;=\; -\,kT \log 2 \;+\; \frac{1}{2}kT \log\left(1 - M^2\right) \;+\; \frac{1}{2}qJM^2\,, \tag{2}$$

By assuming the expansion $M^2 = Ct + Dt^2 + \ldots$, where for small $t \equiv (T - T_c)/T_c$, derive that

$$A \;=\; -\,kT \log 2 \;+\; \frac{1}{2}kT \log\left(1 - Ct - Dt^2 - \ldots\right) \;+\; \frac{1}{2}qJ(Ct + Dt^2 + \ldots)\,.$$

By expanding A in t show that

$$A \;=\; a_0 + a_1 t + \frac{1}{2}a_2 t^2 + \ldots\,,$$

for constants a_i, and consequently that

$$C \;=\; a_2 \;+\; O(t),$$

and hence that the exponent α is given by $\alpha = 0$.

We note also that, except for α we can get all exponents from the equation of state by expanding tanh near $t = 0$:

$$M \;=\; (\beta qJM + \beta h) \;-\; \frac{1}{3}\;(\beta qJM + \beta h)^3 \;+\; \ldots\,,$$

or, keeping only linear terms in h (ignore e.g. hM^2 since this gives non-leading singular behaviour – check it),

$$\frac{1}{3}(\beta qJ)^3 M^3 \;+\; (1 - \beta qJ)M \;-\; \beta h \;=\; 0\,,$$

as we find from minimizing A.

(2) In the Blume-Capel model in D-dimensions the spins σ_n take values $\sigma_n = 1, 0, -1$. The Hamiltonian is an extension of the Ising-like one discussed in the previous question:

$$H \;=\; -J\sum_n \sigma_n\sigma_{n+\mu} \;+\; g\sum_n \sigma_n^2 \;-\; h\sum_n \sigma_n \;.$$

Use the mean field approach to show that the free energy of this system is approximated by

$$A \;\equiv\; \frac{F}{N} \;=\; \frac{1}{2}Jq\,M^2 - kT\;\log\left(1 + 2\kappa\,\cosh\beta(JqM + h)\right)\;.$$

where $\kappa = \exp(-\beta g)$. (*Hint: do not approximate the* $g\sum_\sigma \sigma_n^2$ *term.*)

For $h = 0$ expand A as a power series in M. For what values of (T, κ) does mean field theory predict (i) ordinary critical behaviour, (ii) tricritical behaviour, (iii) a first order transition? In each case find the value of the critical temperature $T_c(\kappa)$.

Calculate the critical exponent α for both critical and tricritical behaviours.

(3) The free energy, A, of an Ising system with order parameter, M, is given by

$$A = -hM + A_2M^2 + A_4M^4 + A_6M^6,$$

Q2

where it is assumed that $A_6 > 0$ and that A_2 and A_4 are functions of the external fields T and g, with $A_2 \sim (T - T_c(g))$ and where h is the applied magnetic field. (Note, this is similar to Q8 where $\kappa = \exp(-\beta g)$.)

On dimensional grounds argue that at equilibrium A may be expressed as

$$A = \frac{|A_2|^{\frac{3}{2}}}{A_6^{\frac{1}{2}}}F\left(\frac{A_4}{2|A_2|^{\frac{1}{2}}A_6^{\frac{1}{2}}}, \frac{hA_6^{\frac{1}{4}}}{|A_2|^{\frac{5}{4}}}\right),$$

where $F(0,0)$ is finite and non-zero.

Compare this expression with the **generic** form for the free energy, A, near the tri-critical point, namely

$$A = |T - T_c(\tilde{g})|^{2-\alpha}F\left(\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}}, \frac{h}{|T - T_c(\tilde{g})|^{\Delta}}\right),$$

where $\tilde{g} \propto A_4$ and $\tilde{g}$ has been substituted for g as one of the independent external fields. Deduce that

$$\alpha = \frac{1}{2}, \quad \phi = \frac{1}{2}, \quad \Delta = \frac{5}{4}.$$

Define the critical temperature at the tricritical point to be $T_{TCP} \equiv T_c(\tilde{g} = 0)$.

(i) For $h = 0$ consider the trajectory in $(T, \tilde{g})$ space defined by the limit

$$\tilde{g} \to 0, \qquad T \to T_{TCP},$$

with

$$\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}} = x \qquad \text{fixed.}$$

Observe that $A \sim |T - T_{TCP}|^{\frac{3}{2}} \;\Rightarrow \alpha = \frac{1}{2}$, The trajectory lies in the **tricritical region**, i.e., we see tricritical exponents as we approach the transition. ϕ is known as the crossover exponent since it controls the shape of the trajectory nd hence defines the boundary of the tricritical region.

(ii) For $h = 0$, $\tilde{g}$ fixed and $T \to T_c$ show that $\alpha = 0$ (i.e. normal critical behaviour) as long as it can be assumed that the function G defined by

$$yG(y, 0) = F(\frac{1}{y}, 0)$$

is finite and non-zero at $y = 0$.

The crucial point is that to use dimensional analysis the existence of scaling functions such as F and G must be assumed and that these functions are finite and non-zero when their arguments are set to zero.

The Renormalization Group

(4) Derive the RG transformation equations for the 1D Ising model as given in the notes:

$$\begin{aligned} x' &= \frac{x(1+y)^2}{(x+y)(1+xy)} \\ y' &= \frac{y(x+y)}{(1+xy)} \\ w' &= \frac{w^2xy^2}{(1+y)^2(x+y)(1+xy)} . \end{aligned}$$

where

$$x = e^{-4\beta J}, \quad y = e^{-2\beta h}, \quad w = e^{4\beta C} .$$

Show that there is a fixed point at $x = 0, y = 1$. Linearize the transformation about this fixed point, derive the fixed point eigenvalues and the two associated critical indices. Hence, deduce that the singular part of the free energy per spin satisfies

$$f(x, \rho) = b^{-1} f(b^2 x, b\rho),$$

for a scale change $b = 2^p$ and where $y = 1 - \rho$. Use this result to show that

$$f(x, \rho) = \sqrt{x} \tilde{f}(\rho/\sqrt{x}) ,$$

where $\tilde{f}(z) = f(1, z)$. Verify that this is consistent with the exact result for the free energy if we choose

$$\tilde{f}(z) = -kT\sqrt{1 + z^2/4} .$$

for the **singular** part.

(5) Compare the complete expression for the free energy F, derived in the notes, with its scaling form. What plays the rôle of the inhomogeneous part?

For fixed J and $h = 0$ find an expression as $T \to 0$ for the leading singularity in the analogue of the specific heat $C = \partial^2 F/\partial t^2$. Comment on what this implies for the value of α and the validity of the scaling relation $\alpha + 2\beta + \gamma = 2$.

Suppose now $T > 0$ and fixed with $h = 0$ and let $J \to \infty$. What is the value of α in this case?

(6) (*This is Q5.2 from Binney et al. "The Theory of Critical Phenomena"*) Consider the two-dimensional Ising model on a square lattice with nearest- and next-nearest-neighbour interactions only, and denote the couplings by K and L, respectively. Perform a thinning of degrees of freedom by summing over the spins on every second

site in a 'checker-board' fashion. Rescale the lattice by factor $b = \sqrt{2}$ to recover the original lattice spacing. Calculate the interactions on the blocked lattice keeping only terms up to $O(K^2)$ and $O(L)$. Show that there are only two such interactions to this order which are the same operators as the original ones but with couplings

$$K' = 2K^2 + L, \qquad L' = K^2 .$$

Find the critical points for these RG equations and identify the non-trivial one. Linearizing about this point, find a value for the exponent ν.

Sketch the RG flows in the K, L plane with $K > 0, L > 0$.

(1) This is book work see p 23

$$Z = e^{-\frac{1}{2}\beta q N M^2} \sum_{\sigma} e^{\beta(qJM+h)\sum_{n}\sigma_{n}}$$

This first factor does not depend on σ and so can be ignored in expectation values. Also, can write Z as

$$Z \sim \prod_{n} \left(\sum_{\sigma_n} e^{\beta(qJM+h)\sigma_n} \right)$$

ie it factors into a product. Thus

$$M = \langle \sigma_0 \rangle = \frac{\sum_{\sigma} \sigma e^{\beta(qJM+h)\sigma}}{\sum_{\sigma} e^{\beta(qJM+h)\sigma}}$$

or

$$\underline{\underline{M = \tanh \beta(qJM+h)}}$$

$$A = -kT \log\left[2\cosh\beta(qJM+h)\right] + \tfrac{1}{2} qJM^2$$

$$\frac{\partial A}{\partial M} = 0 \quad \Rightarrow$$

$$- kT \tanh \beta(qJM + h) \cdot \beta q J$$
$$+ qJM = 0$$

or $\underline{\underline{M = \tanh \beta(qJM + h)}}$

$h = 0$ & $\cosh x = (1 - \tanh^2 x)^{-1/2}$

$$A = -kT \log 2 + kT \tfrac{1}{2} \log(1 - M^2)$$
$$+ \tfrac{1}{2} qJM^2$$

(not specific heat)

$$M^2 = Ct + Dt^2 + \dots \Rightarrow$$

$$A = -kT \log 2 + \tfrac{1}{2} kT \left(-M^2 - \frac{M^4}{2} - \dots\right)$$
$$+ \tfrac{1}{2} qJM^2$$

$$= -kT \log 2 - \tfrac{1}{2} kT \left(Ct + Dt^2 + \frac{C^2 t^2}{2}\right)$$
$$+ \tfrac{1}{2} qJ (Ct + Dt^2 \dots)$$

Hence can write

$$A = a_0 + a_1 t + \tfrac{1}{2} a_2 t^2 + \dots$$

$\Rightarrow$ Specific heat $C = a_2 + O(t)$

$\Rightarrow$ $\underline{\underline{\alpha = 0}}$

2) Blume-Capel model in MFT $S_n = 1, 0, -1$.

$$\mathcal{H} = -J\sum S_n S_{n+\mu} + g\sum S_n^2 - h\sum S_n$$

$$H_{MF} = \frac{1}{2}JqNM^2 - JqM\sum S_n + g\sum S_n^2 - h\sum S_n$$

$\Rightarrow$

$$Z = e^{-\beta\frac{1}{2}JqNM^2} \cdot \prod_n \sum_{S_n} e^{\beta[(JqM+h)S_n - \Delta S_n^2]}$$

(see p23 eqn 38)

$\Rightarrow$

$$F = \frac{1}{2}JqNM^2 - kTN\log\left(1 + 2e^{-\beta g}\cosh[\beta(JqM+h)]\right)$$

$\Rightarrow$

$$\frac{F}{N} = \frac{1}{2}JqM^2 - kT\log\left(1 + K\cosh[\beta(JqM+h)]\right)$$

where $K = 2e^{-\beta g}$ ($K = e^{-\beta g}$ in Q but just redefinition of K)

set $\underline{h=0}$ now $\Rightarrow$

$$\frac{F}{N} = \frac{1}{2}JqM^2$$
$$- kT\log\left(1 + K\left(1 + \frac{(\alpha M)^2}{2} + \frac{1}{24}(\alpha M)^4 + \ldots\right)\right)$$

where $\alpha = \beta Jq$

Expand log

$$\frac{F}{N} = -kT\log(1+K) + \frac{1}{2}JqM^2$$
$$- kT\log\left(1 + \frac{K}{1+K}\left(\frac{1}{2}\alpha^2M^2 + \frac{1}{24}\alpha^4M^4 + \ldots\right)\right)$$

Coeff M^2:

$$-\frac{kTK\alpha^2}{2(1+K)} + \frac{1}{2}Jq$$

Coeff M^4:

$$-kT\left[\frac{K\alpha^4}{24(1+K)} - \frac{K\alpha^4}{8(1+K)^2}\right]$$

$$= \frac{-kTK\alpha^4}{24(1+K)^2}(1+K-3K)$$

$$= -\frac{kTK\alpha^4}{24(1+K)^2}(1-2K)$$

$$A_4 \geq 0 \Rightarrow 2K \geq 1 \Rightarrow \underline{K \geq 1/2}$$

$$\Rightarrow \underline{\underline{e^{-\beta g} \geq \frac{1}{2}}}$$

$$A_2 = \frac{Jq}{T}\Big[T - \underbrace{\frac{KJq}{1+K}}_{T_c}\Big]$$

$$T_{TCP} = T_c \text{ at } K = \frac{1}{2}$$

$$\Rightarrow T_{TCP} = \frac{Jq}{3}.$$

Since TCP at $\underline{\underline{T = T_0 = Jq/3 \quad K = 1/2}}$

$$\Rightarrow \underline{g = \frac{Jq}{3}\log 2}$$

3) $[A_n] = -(D + n D_M)$

Have $h \equiv A_1$, $A_2 \propto (T - T_c)$, A_4, A_6.

$A_6 > 0$ and stabilized free energy at large M

Want $[A] = -D$ and form this from A_2 & A_6. Unique combination is

$$\underline{\underline{\frac{|A_2|^{3/2}}{A_6^{1/2}}}}$$

Dimensionless variables

* $h = 0$ form from A_2, A_4, A_6. Can choose coeff A_4 to appear linearly. Then get

$$\underline{\underline{\frac{A_4}{2|A_2|^{1/2} A_6^{1/2}}}}$$ (factor of 2 seems to be convention)

* $h \neq 0$ form from h, A_6, A_2 and choose h to appear linearly. Get

$$\underline{\underline{\frac{h A_6^{1/4}}{|A_2|^{5/4}}}}$$

Can then write

$$A = \frac{|A_2|^{3/2}}{A_6^{1/2}} F\left(\frac{A_4}{2|A_2|^{1/2} A_6^{1/2}}, \frac{h A_6^{1/4}}{|A_2|^{5/4}} \right)$$

Generic scaling form is

$$A = |T - T_c(\tilde{g})|^{2-\alpha} F\left(\frac{\tilde{g}}{|T-T_c(\tilde{g})|^{\phi}}, \frac{h}{|T-T_c(\tilde{g})|^{\Delta}} \right)$$

$$A_4 \Leftrightarrow \tilde{g}$$

We read the LG values

$$\alpha = \tfrac{1}{2}, \quad \phi = \tfrac{1}{2}, \quad \Delta = 5/4$$

$T_{TCP} = T_c(\tilde{g} = 0)$ ie $\underline{A_2 = A_4 = 0}$

(i) $h = 0$ $\quad \tilde{g} \to 0$ $\quad T \to T_{TCP}$ such ~~that~~

$$\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}} = x \text{ fixed}$$

Then

$$A \sim |T - T_{TCP}|^{3/2} F(x, 0)$$

ie $\alpha = 3/2$ and have TC behaviour

See Figure on p2 of following notes

(ii) $h = 0$ $\tilde{g}$ fixed $T \to T_c(\tilde{g})$

must rearrange:

$$A = \frac{|A_2|^{3/2}}{A_6^{1/2}} \frac{2|A_2|^{1/2} A_6^{1/2}}{A_4} G\left(\frac{2|A_2|^{1/2} A_6^{1/2}}{A_4}, 0\right)$$

$$= \frac{2|A_2|^2}{A_4} \hat{G}\left(\frac{|A_2|^{1/2}}{A_4}\right)$$

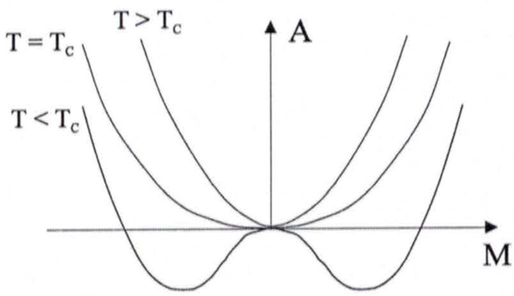
ie ordinary critical behaviour as long as $\hat{G}(0) \neq 0$ & finite.

ϕ is called the cross-over exponent

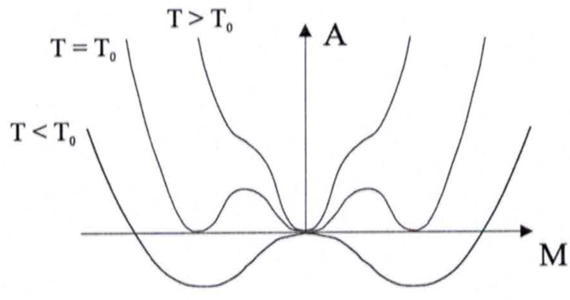
Here $y = \dfrac{2|A_2|^{1/2} A_6^{1/2}}{A_4}$ so

$$A = \frac{|A_2|^{3/2}}{A_6^{1/2}} y \cdot \underbrace{\frac{1}{y} F(1/y, 0)}_{G(y,0)}$$

$A_4 \leq 0$
As T decreases $A(M)$ behaves qualitatively differently depending on whether $A_4 > 0$ or $A_4 \leq 0$:



second-order transition at $T = T_c$



first-order phase transition at T_0, $T_0 > T_c$

Hence the system passes from a second-order transition to a first-order transition as A_4 changes sign and becomes negative.
The stationary points are at $M = 0$ and at

$$M^2 = \left[-A_4 \pm \left(A_4^2 - 4A_2A_6\right)^{\frac{1}{2}}\right] / 2A_6 \equiv M_\pm^2.$$

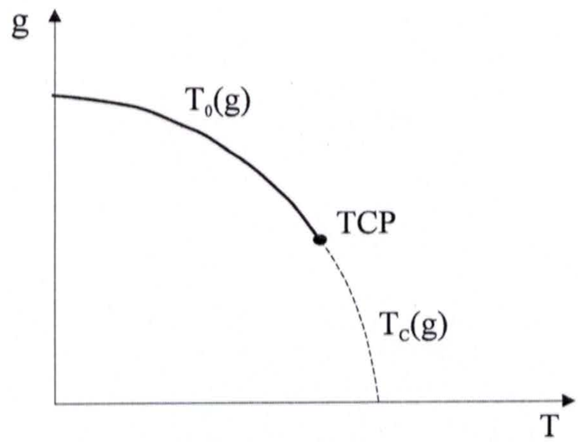
The $+$ sign gives the minima and the $-$ sign the maxima.
T_0 is determined by $A(M) = 0$ having a double root at $M = \pm M_+$ (note that A_0 is set to zero so that $A(0) = 0$ is the minimum for $T > T_0$). The solution is

$$A_2 = \frac{3}{16}\frac{A_4^2}{A_6}, \tag{1}$$

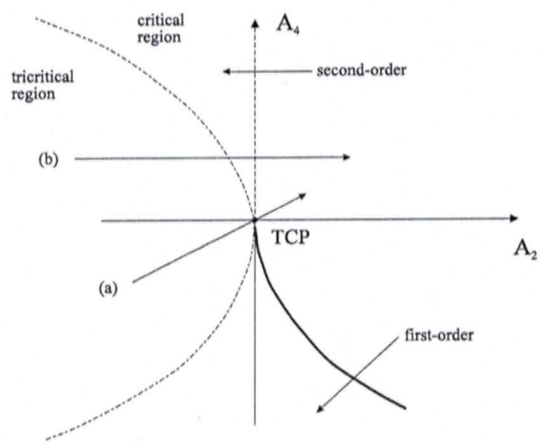
and at the transition

$$\left(\Delta M^2\right)^2 = M_+^2 = -\frac{3}{4}\frac{A_4}{A_6} \tag{2}$$

Thus the point $T = T_c$, $A_4 = 0$ separates the first-order line from the second-order line: this is a **tricritical point**. To see the tricritical point these two parameters have to take these special values and this requires tuning two external fields in the phase diagram.
In the space of physical fields, denoted by T and g (e.g., g can be identified with a chemical potential controlling the relative abundances in a two component system), the phase diagram has the form:



Alternatively the phase diagram in (A_2, A_4) space takes the form:



(i) **Both** trajectories (a) and (b) exhibit a second-order transition.

(ii) Trajectory (a) passes through the TCP and lies entirely within the tricritical region. The transition is characterised entirely by the properties of the TCP, and all critical exponents are tricritical ones.

(iii) Trajectory (b) exhibits an **ordinary** second order transition. However, it starts in the tricritical region and so initially the divergence of the relevant quantities is controlled by the TCP. Eventually it passes into the critical region and the transition is characterised by the line of ordinary critical points and the critical exponents that are given above.

In other words we only see a transition controlled by the TCP when we apporoach along a trajectory lying in the tricritical region. For trajectories that pass from one region to another we see a change in the critical behaiviour. This change is characterised by **crossover exponents**.

At the TCP $A_4 = 0$, and so we have

$$|M| = \left|\frac{A_2}{A_6}\right|^{\beta_t} \quad \text{with} \quad \beta_t = \frac{1}{4}.$$

New exponents can be defined, see Eqns (1),(2):

$$\begin{aligned}(T_0 - T_c) &\sim (-A_4)^{\frac{1}{\psi}} \quad \text{with} \quad \psi = \frac{1}{2} \\ \Delta M &\sim |A_4|^{\beta_u} \quad \text{with} \quad \beta_u = \frac{1}{2},\end{aligned}$$

where ΔM is the discontinuity in M across the first-order line ($A_4 < 0$). then for small A_4 we can write

$$\begin{aligned}|M| &= \left|\frac{A_2}{A_6}\right|^{\beta_t} m(x), \\ x &= \frac{A_4}{2|A_2|^{\frac{1}{2}} A_6^{\frac{1}{2}}}\end{aligned}$$

where x is a dimensionless variable which vanishes as $A_4 \to 0$. Note that A_2 remains **non-zero** at a TCP. To see tricritical behaviour along a trajectory we clearly need x small, i.e.,

$$A_4^2 \ll 4|A_2|A_6.$$

This defines a parabola in the (A_4, A_6) plane separating the tricritical from critical regions. This is shown on the figure. In the space of physical parameters it translates into a similar shaped curve defining the two regions controlled by the TCP and ordinary critical points respectively.

The general theory of continuous phase transitions can be encoded in terms of scaling functions and relies on dimensional analysis together with some assumptions about the behaviour of the scaling functions for small argument. If naive or engineering dimensions are used this is generally a recoding of Landau theory but is often used to describe the behaviour of the relevant thermodynamic variables as a function of the actual external fields and hence parametrises the experimental observations.
Add a magnetic field, h, with contribution to the free energy of $-hM$. Then we can always write

$$A = \frac{|A_2|^{\frac{3}{2}}}{A_6^{\frac{1}{2}}} F\left(\frac{A_4}{2|A_2|^{\frac{1}{2}} A_6^{\frac{1}{2}}}, \frac{hA_6^{\frac{1}{4}}}{|A_2|^{\frac{5}{4}}}\right).$$

The point is that the **equilibrium** free energy, A, can always be written in terms of dimensionless ratios in this way. As before assign dimension (-1) to M and dimension d to A, and then A_n has dimension $(d{+}n)$. The above expression is then a general way of writing the dependence of A at equilibrium on the coefficients

A_{2n} in terms of a **scaling function**, F. Note that since A_6 is always taken as positive it causes no problem in the denominators.
We now compare with a standard form parametrising A near the TCP:

$$A = |T - T_c(\tilde{g})|^{2-\alpha}\ F\left(\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}}, \frac{h}{|T - T_c(\tilde{g})|^{\Delta}}\right),$$

where $\tilde{g} \propto A_4$, and thus measures the distance from the TCP along the tangent to the critical line at the TCP. Note that $\tilde{g}$ has been substituted for the field g as the second independent external field: the critical line is thus parametrised as $T_c(\tilde{g})$. The TCP then is at position $(0, T_t)$ in the $(\tilde{g}, T)$ plane where $T_t = T_c(0)$. Labelling the critical exponents at the TCP by suffix, t, we clearly have

$$\alpha_t = \frac{1}{2}, \quad \phi_t = \frac{1}{2}, \quad \Delta_t = \frac{5}{4}.$$

The following examples clarify the interpretation.

(i) $h = 0,\ \tilde{g} \to 0,\ T \to T_t$ such that $\frac{\tilde{g}}{|T - T_t|^{\phi_t}} \equiv x$ is fixed. Then

$$A = |T - T_t|^{\frac{3}{2}} F(x, 0) \quad \text{with} \quad F(0,0) \quad \textbf{finite}.$$

We see tricritical behaviour and since $\tilde{g} \sim |T - T_t|^{\phi_t}$ the trajectory lies in the tricritical region. ϕ_t is the **cross-over exponent**.

(ii) $h \to 0,\ \tilde{g}$ fixed, $T \to T_c$.

$$A = |T - T_c|^{\frac{3}{2}}\ F\left(\frac{\tilde{g}}{|T - T_c|^{\frac{1}{2}}}, 0\right).$$

the argument of F is not under control and so we rearrange the expression:

$$A = \frac{|T - T_c|^2}{\tilde{g}}\ G\left(\frac{|T - T_c|^{\frac{1}{2}}}{\tilde{g}}, 0\right),$$

where $G(z, 0) = zF(\frac{1}{z}, 0)$ and $G(0,0)$ is finite and non-zero. This property of G is an assumption in the general theory and could be violated. It **does** follow from the standard Landau analysis and hence if it turned out to be false in an experiment it would signal a breakdown of the Landau theory. The goal then would be to rescue the dimensional analysis approach by assigning values to the dimensions of the parameters different from the naive ones but which render the scaling functions F and G well behaved for small argument.

In this case we find that A shows the normal critical behaviour associated with an ordinary critical point, namely

$$A \sim |T - T_c|^{2-\alpha} \quad \text{with} \quad \alpha = 0.$$

4) $x = e^{-4\beta J}$ $y = e^{-2\beta h}$ $w = e^{4\beta C}$

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)}$$

$$y' = \frac{y(x+y)}{(1+xy)}$$

$$w' = \frac{w^2 x y^2}{(1+y)^2 (x+y)(1+xy)}$$

Let

$z = e^{\beta J}$ $\mu = e^{\beta h}$ $c = e^{-\beta C}$

$\Rightarrow$

$x = z^{-4}$ $y = \mu^{-2}$ $w = c^{-4}$

The RG equation is $W' = W^2 \Rightarrow$

$$c' \begin{pmatrix} \mu' z' & z'^{-1} \\ z'^{-1} & \mu'^{-1} z' \end{pmatrix} = c^2 \begin{pmatrix} \mu z & z^{-1} \\ z^{-1} & \mu^{-1} z \end{pmatrix}^2$$

$\Rightarrow$

1) $c' \mu' z' = c^2 (\mu^2 z^2 + z^{-2})$

2) $c' z'^{-1} = c^2 (\mu + \mu^{-1})$

3) $c' \mu'^{-1} z' = c^2 (z^{-2} + \mu^{-2} z^2)$

3)/1)

$$\mu'^{-2} = \frac{z^{-2} + \mu^{-2} z^{2}}{z^{-2} + \mu^{2} z^{-2}}$$

$\Rightarrow$

$$y' = \frac{z^{-4} + \mu^{-2}}{z^{-4} + \mu^{2}}$$

$\Rightarrow$

$$y' = \frac{x + y}{x + 1/y}$$

$\Rightarrow$

$$y' = \frac{y(x+y)}{(1+xy)}$$

2)/3)

$$z'^{-2} \mu' = \frac{\mu + \mu^{-1}}{\mu^{2} z^{2} + z^{-2}}$$

$\Rightarrow$

$$z'^{-4} = \mu'^{-2} \frac{(\mu^{2} + \mu^{-2} + 2) z^{-4}}{(\mu^{-2} + z^{-4})^{2}}$$

$\Rightarrow$

$$x' = y' \frac{(1/y + y + 2) x}{(y + x)^{2}}$$

$\Rightarrow$

$$x' = \frac{x(1+y)^{2}}{(x+y)(1+xy)}$$

(2)x(3)

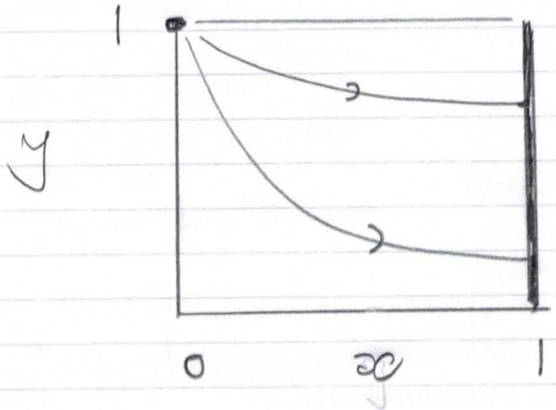
$$c'^2 \mu'^{-1} = \frac{c^4 (\mu + \mu^{-1})}{(\mu^{-2} + z^{-4}) z^2}$$

$\Rightarrow$

$$w' = \frac{y' \, w^2 \, x}{\frac{1}{y} (1+y)^2 (y+x)^2}$$

$\Rightarrow$

$$w' = \frac{w^2 \, xy^2}{(1+y)^2 (x+y)(1+xy)}$$



Let $x = \epsilon \quad y = 1 - \rho$

$$\epsilon' = \frac{\epsilon (2-\rho)^2}{(1-\rho+\epsilon)(1+\epsilon-\epsilon\rho)}$$

$\epsilon' = 4\epsilon$ + higher order

$$1 - p' = \frac{(1-p)(\epsilon + 1 - p)}{1 + \epsilon - \epsilon p}$$

$$\Rightarrow \quad 1 - p' = (1-p)^2 + \text{higher order}$$

$$\Rightarrow \quad \underline{\underline{p' = 2p}} + \text{higher order}$$

$$x = 1 - \epsilon$$

$$x' - \epsilon' = \frac{(1-\epsilon)(1+y)^2}{(1-\epsilon+y)(1+(1-\epsilon)y)}$$

$$= \frac{(1-\epsilon)(1+y)^2}{(1+y)^2\left(1 - \frac{\epsilon}{1+y}\right)\left(1 - \frac{\epsilon y}{1+y}\right)}$$

$$= (1-\epsilon)\left(1 + \frac{\epsilon}{1+y} + O(\epsilon^2)\right)\left(1 + \frac{\epsilon y}{1+y} + O(\epsilon^2)\right)$$

$$= (1-\epsilon)(1 + \epsilon + O(\epsilon^2))$$

$$\Rightarrow \quad 1 - \epsilon' = 1 - O(\epsilon^2)$$

$$\Rightarrow \quad \underline{\underline{\epsilon' = O(\epsilon^2)}}$$

$$y' = y\frac{(1+y)}{(1+y)} + \ldots$$

$$\Rightarrow \quad \underline{\underline{y' = y}} + \text{higher order}$$

$$f(x, \rho) = b^{-D} f(b^{\lambda_t} x, b^{\lambda_h} \rho)$$

At non-trivial f.p. $\lambda_t = 2$ $\lambda_h = 1$

(Remember $b = 2$ in this case). So

$$f(x, \rho) = b^{-1} f(b^2 x, b\rho)$$

Set $b^2 x = 1$

$$f(x, \rho) = \sqrt{x}\ \bar{f}\,(\rho/\sqrt{x})$$

$$\bar{f}(u) = f(1, u)$$

$$\bar{f}(u) = -kT\,(1 + u^2/4)^{1/2}$$

Show from

$$\lambda_+ = \left[z \cosh B + \sqrt{z^2 \sinh^2 B + z^{-2}}\right]$$

$$z = e^{\beta J} \qquad \mu = e^{\beta h} \qquad B = \beta h$$

$z^{-4} = \mu$ small

$$y = 1 - \rho = e^{-2B} \qquad \rho \text{ small}$$

$$\Rightarrow \quad \rho \sim 2B \quad \Rightarrow \quad B \sim \rho/2$$

$$\lambda_+ = z\left[1 + \sqrt{B^2 + z^{-4}}\right]$$

In this approximation then have

$\Rightarrow$

$$\lambda_+ = z\left[1 + \sqrt{x + \rho^2/4}\right]$$

$$\frac{F}{N} = -\frac{1}{\beta}\log\lambda_+ \qquad \log z = \beta J$$

$$\Rightarrow \quad \frac{F}{N} = \underbrace{-J}_{\text{inhomogeneous part}} \underbrace{- kT\sqrt{x + \rho^2/4}}_{\text{singular part, } = f}$$

$$\frac{f}{N} = -kT\, x^{\frac{1}{2}}\left(1 + \frac{\rho^2}{4x}\right)^{\frac{1}{2}}$$

$$\equiv \sqrt{x}\;\bar{f}\,(\rho/\sqrt{x})$$

5) From above the inhomogeneous part is $F_{inh} = -J$

Let $t = e^{-4\beta J}$ as in Ising model notes $\Rightarrow$ $t = x$.

(i) For J fixed F_{inh} plays no rôle and so

$$\frac{f}{N} \sim \frac{-4J}{\log t} \, t^{1/2} \, \bar{f}(0)$$

$$\frac{\partial^2 f}{\partial t^2} \sim \frac{1}{t^{3/2} \log^3 t} + \dots$$

$$\sim t^{-3/2} (\log t)^{-3}$$

$\Rightarrow \alpha = 3/2$ with log corrections

Then before $\beta = 0$, $\gamma = 1/2 \Rightarrow$

$\underline{\underline{\alpha + 2\beta + \gamma = 2}}$ + log corrections

(ii) For T fixed and F_{inh} is important.

(a) $\frac{f}{N} \sim t^{\frac{1}{2}} \Rightarrow \frac{\partial^2 f}{\partial t^2} = 0 \, t^{-3/2}$

~~so~~ predicts ~~exp~~ α ~~=~~ $= \underline{3/2}$ maybe

(b) $F_{inh} = \frac{1}{4} kT \log t$

$\Rightarrow \frac{\partial^2 F_{inh}}{\partial t^2} \sim \frac{1}{t^2} \Rightarrow \alpha = 2$

Thus is up to log corrections.
The non-standard behaviour is because the 1D Ising model has $T_c = 0$ which causes pathologies.

From (b) get $\alpha = 2$.

$$\alpha + 2\beta + \gamma = 5/2.$$

This can be discovered by looking at the behaviour of

$$f_{inh} = \sum_{j=0}^{p-1} b^{-j} g(b^2 t)$$

but it is rather long.

6) Decimated sites are "odd"
Remaining " " "even"

$$\mathcal{H} = -K \sum_{\langle ij \rangle} \sigma_i \sigma_j - L \sum_{\langle ij \rangle_e} \sigma_i \sigma_j - L \sum_{\langle ij \rangle_o} \sigma_i \sigma_j$$

$$e^{-\mathcal{H}} = 1 - \mathcal{H} + \tfrac{1}{2} \mathcal{H}^2 + \cdots$$

$$= 1 + K \sum_{\langle ij \rangle} \sigma_i \sigma_j + L \sum_{\langle ij \rangle_e} \sigma_i \sigma_j + L \sum_{\langle ij \rangle_o} \sigma_i \sigma_j + \frac{1}{2} K^2 \sum_{\substack{\langle ij \rangle \\ \langle kl \rangle}} \sigma_i \sigma_j \sigma_k \sigma_l \qquad (*)$$

Have kept up to & including $O(L)$, $O(K^2)$

$$e^{\mathcal{H}(\sigma', K', L')} = \sum_{\sigma} T(\sigma', \sigma) e^{-\mathcal{H}(\sigma, K, L)}$$

Like 1D Ising model the kernel $T(\sigma', \sigma)$ is

$$T(\sigma', \sigma) = \delta_{\sigma_e' \sigma_e}$$

$N \rightarrow N'$ with $N = 2N'$

Thus need RHS

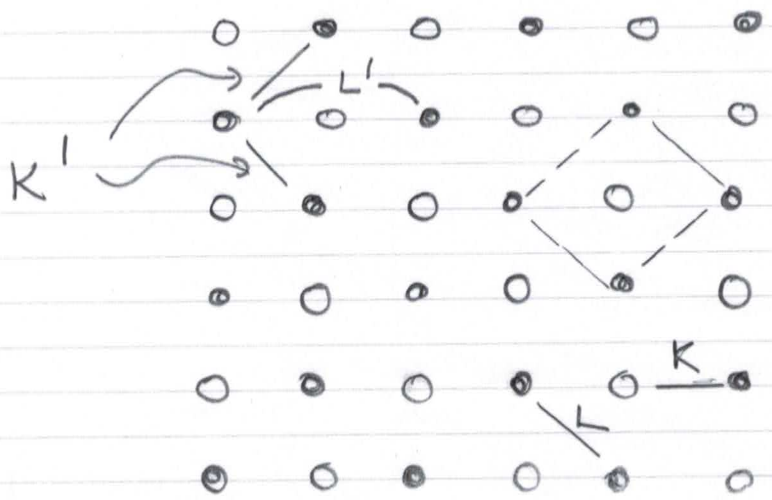
$$\sum_{\{\sigma_0\}} e^{-\mathcal{H}} = \sum_{\{\sigma_0\}} (*)$$

2nd and 4th term in (*) sum to zero since have equal numbers of +1 & −1.

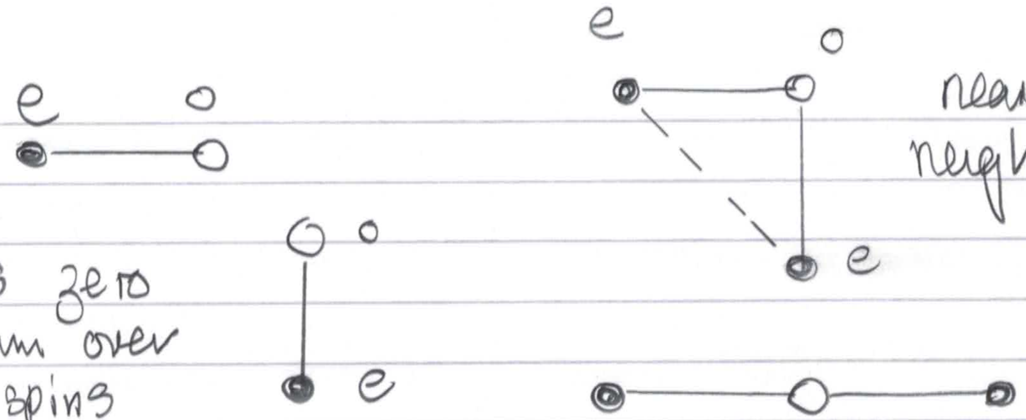
Look at last term

$$\frac{1}{2}K^2 \sum_{\substack{\{\sigma_0\} \\ \langle ij\rangle \\ \langle kl\rangle}} \sigma_i\sigma_j\sigma_k\sigma_l$$

These terms will cancel unless same "odd" spin included twice.



Blocked lattice of "•" at 45° to original lattice $a \to \sqrt{2}a$



gives zero on sum over odd spins

nearest neighbour

next-nearest neighbour

* There are two ways for nearest neighbour <u>and</u> two orderings $\langle i,j\rangle \leftrightarrow \langle k,l\rangle$. Get:

$$2^{N'}\, 2K^2 \sum_{\langle ij\rangle'} \sigma_i' \sigma_j'$$

Note $2^{N'}$ from $\sum_{\{\sigma_0\}}$ e.g.

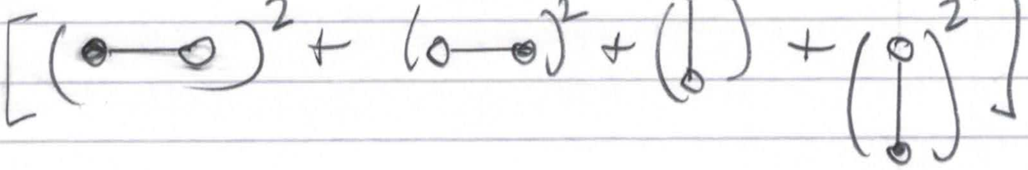
$\sum_{\{\sigma_0\}} 1 = 2^{N'}$. This occurs in all terms.

* There is one way for next nearest neighbour <u>and</u> two $\langle i,j\rangle \leftrightarrow \langle kl\rangle$ orderings:

$$2^{N'}\, K^2 \sum_{\langle ij\rangle'_{nn}} \sigma_i \sigma_j$$

* $2^{N'}$ from "1"

* $2^{N'}\, 2K^2$ from last term

$\frac{1}{2} K^2 \sum_{\{\sigma_0\}}$ 

Thus get

$$e^{-H'(\sigma, K', L')} = 2^{N'} + 2^{N'} 2K^2$$

$$+ 2^{N'} (L + 2K^2) \sum_{\langle ij \rangle'} \sigma_i \sigma_j$$

$$+ 2^{N'} K^2 \sum_{\langle ij \rangle'_{nn}} \sigma_i \sigma_j$$

take $-\log \Rightarrow$

$$H' = N' \log 2 -$$

$$\log \left(1 + 2K^2 + (L + 2K^2) \sum_{\langle ij \rangle'} \sigma_i \sigma_j \right.$$

$$\left. + K^2 \sum_{\langle ij \rangle'_{nn}} \sigma_i \sigma_j \right)$$

Constants accumulate into inhomog. part

Then find

$$K' = L + 2K^2, \quad L' = K^2$$

Non-trivial f.p. at

$$K^* = \frac{1}{3} \quad L^* = \frac{1}{9}$$

Linearize RGE

$$\delta K' = \delta L + 4 \cdot \tfrac{1}{3} \delta K$$

$$\delta L' = 2 \cdot \tfrac{1}{3} \delta K$$

$$\begin{pmatrix} \delta K' \\ \delta L' \end{pmatrix} = \begin{pmatrix} 4/3 & 1 \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} \delta K \\ \delta L \end{pmatrix}$$

Eigenvalues $\lambda_{\pm} = \frac{1}{3}(2 \pm \sqrt{10})$

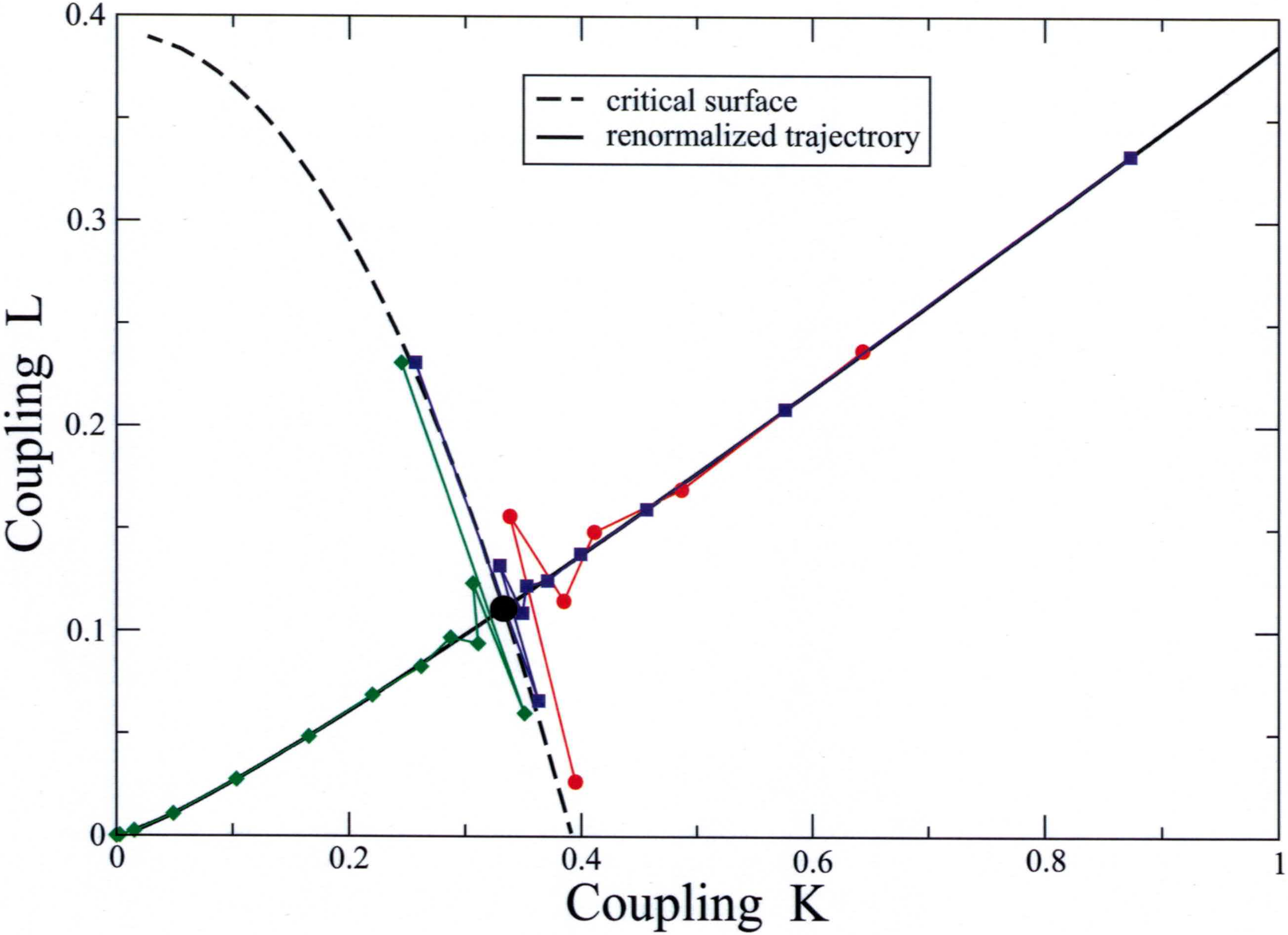
$|\lambda_-| < 1 \Rightarrow$ irrelevant

$$\lambda_+ = 1.720\ldots$$

Exponent is

$$\nu = \frac{\log \lambda_+}{\log \sqrt{2}} \qquad (a' = \sqrt{2}\, a)$$

$$\Rightarrow \underline{\underline{\nu = 0.638\ldots}}$$


critical surface
renormalized trajectrory
Coupling L
Coupling K
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1