**Examples 1**

*Thermodynamics*

(1) In standard notation, the first law of thermodynamics for a magnetic system is

\[ dU = TdS - MdH . \]

where \( U \) is the internal energy, \( M \) is the magnetization and \( H \) the applied magnetic field. Define the other functions of state by appropriate Legendre transformations, for example \( F = U - TS \) and hence obtain the Maxwell relations:

\[
\begin{align*}
\left( \frac{\partial T}{\partial H} \right)_S &= -\left( \frac{\partial M}{\partial S} \right)_H, \\
\left( \frac{\partial S}{\partial M} \right)_T &= -\left( \frac{\partial H}{\partial T} \right)_M, \\
\left( \frac{\partial S}{\partial H} \right)_T &= \left( \frac{\partial M}{\partial T} \right)_H, \\
\left( \frac{\partial T}{\partial M} \right)_S &= \left( \frac{\partial H}{\partial S} \right)_M.
\end{align*}
\]

(2) The specific heats at constant magnetic field and at constant magnetization for the magnetic system are

\[ C_H = T \left( \frac{\partial S}{\partial T} \right)_H, \quad C_M = T \left( \frac{\partial S}{\partial T} \right)_M . \]

The isothermal and adiabatic susceptibilities are

\[ \chi_T = \left( \frac{\partial M}{\partial H} \right)_T, \quad \chi_S = \left( \frac{\partial M}{\partial H} \right)_S . \]

Also define

\[ \alpha_H = \left( \frac{\partial M}{\partial T} \right)_H . \]

From the identity

\[ \left( \frac{\partial S}{\partial T} \right)_M = \left( \frac{\partial S}{\partial T} \right)_H + \left( \frac{\partial S}{\partial H} \right)_T \left( \frac{\partial H}{\partial T} \right)_M, \]

deduce that

\[ \chi_T \left( C_H - C_M \right) = -T \left( \frac{\partial M}{\partial H} \right)_T \left( \frac{\partial M}{\partial T} \right)_H \left( \frac{\partial H}{\partial T} \right)_M, \]

and hence show that

\[ \chi_T \left( C_H - C_M \right) = T \alpha^2_H \quad (†). \]

By similar means show that

\[ C_H (\chi_T - \chi_S) = T \alpha^2_H. \]

Hence show that

\[ \chi_S C_H = \chi_T C_M . \]
(3) For $T \to T_c$ for $T < T_c$ and $H = 0$ the dependence of the following observables on $T$ is parametrized by

$$
C_H \sim (T_c - T)^{-\alpha}, \quad M \sim (T_c - T)^{\beta}, \quad \chi_T \sim (T_c - T)^{-\gamma}.
$$

Using that $C_M > 0$ and (†) above derive Rushbrooke's inequality:

$$
\alpha + 2\beta + \gamma \geq 2.
$$

What is the equivalent set of statements for a gaseous system?

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**Statistical Models and Landau Theory**

(4) The Hamiltonian for a set of $N$ spins $\{s_n\}$ that are 3-dimensional vectors in the presence of a magnetic field $\mathbf{H}$ is

$$
\mathcal{H} = \sum_n \mu H \cos \theta_n,
$$

where $\theta_n$ is the angle between between $s_n$ and $\mathbf{H}$. Show that the partition function is

$$
Z = \left[ 4\pi \left( \frac{\sinh \beta \mu H}{\beta \mu H} \right) \right]^N.
$$

Compute the free energy $F$, the entropy $S$ and the internal energy $U$. Find the equation of state and compute the susceptibility $\chi_T$. Examine the behaviour of $\chi_T$ at low $T$.

(5) In a modification of the 1-dimensional Ising model the spins can take the values $\sigma_n = 1, 0, -1$. Show that the partition function is

$$
Z = \text{Tr} \, W^n,
$$

where $W$ is the $3 \times 3$ matrix

$$
\begin{pmatrix}
z\mu^2 & \mu & z^{-1} \\
\mu & 1 & \mu^{-1} \\
z^{-1} & \mu^{-1} & z\mu^{-2}
\end{pmatrix}
$$

with $z = e^{\beta J}$ and $\mu = e^{\beta h/2}$.

For the case $h = 0$ show that this matrix can be expressed in the form $W = \Lambda \Lambda^{-1}$ where

$$
\Lambda = \begin{pmatrix}
2 \cosh \beta J & \sqrt{2} & 0 \\
\sqrt{2} & 1 & 0 \\
0 & 0 & 2 \sinh \beta J
\end{pmatrix},
\quad
P = \begin{pmatrix}
1/\sqrt{2} & 0 & -1/\sqrt{2} \\
0 & 1 & 0 \\
1/\sqrt{2} & 0 & 1/\sqrt{2}
\end{pmatrix}.
$$
Hence find the eigenvalues of $W$ and show that in the thermodynamic limit the free energy of the system is

$$F = -NkT \log \left\{ \left(1 + 2 \cosh \beta J + \sqrt{(2 \cosh \beta J - 1)^2 + 8}\right)/2 \right\}.$$  

(6) Give a plausible argument that the phase diagram of the 3D spin model with Hamiltonian

$$H = -J \sum_{<ij>} s_i \cdot s_j + \frac{1}{2} g \sum_i \left( (s_i^z)^2 - \frac{1}{2} \left( (s_i^x)^2 + (s_i^y)^2 \right) \right)$$

has the form

![Phase Diagram](image)

where $-\infty < g < \infty$, $<i,j>$ means nearest neighbour pairs, and $s_i$ is a vector at the $i$-th site with $|s_i| = 1$.

You should consider the nature of the dominant configurations for low and high temperature for $|g|$ very large, and the type of transition that is likely to separate them. Then ask what happens for low $T$ as $g$ changes sign. Note that in $D = 3$ the $O(2)$, plane rotator, model (the spin at each site, $s_i$, is a unit vector lying in the $xy$-plane) exhibits a continuous phase transition.

*If you look in the extra material at the end of the notes on the web then you will see the answer.*

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