Example 3

Field Theory

(1) In the Gaussian model with $\alpha = 1$

$$\Delta(p) = \frac{1}{p^2 + m^2}.$$  

Let $x = (s, x)$ where $x$ is a $D - 1$ dimensional vector. Show that

$$\int d^{D-1}x \Delta(s, x) \propto \xi e^{-|s|/\xi},$$

where $\xi = 1/m$ is the correlation length.

An optional and much harder result to obtain is

$$G(r) = \int \frac{d^D p}{(2\pi)^D} e^{-ip \cdot x} \frac{\alpha}{p^2 + \alpha m^2} \sim \begin{cases} \frac{\xi e^{-r/\xi}}{(r\xi)^{(D-1)/2}} & a \ll \xi \ll r, \\ \frac{1}{r^{(D-2)}} & a \ll r \ll \xi, \end{cases}$$

with $\xi^{-2} = \alpha m^2$ and $\alpha$ constant. A hint is to heavily approximate the integral in the two regions by recognizing that for $a \ll \xi \ll r$ it is dominated by $p \ll m$ and for $a \ll r \ll \xi$ it is dominated by $p \gg m$.

In an interacting theory $\alpha$ depends on $p$ and we write

$$\frac{\alpha(p, \xi)}{p^2 + \alpha(p, \xi)m^2} = \frac{Z(p, \xi)}{p^2 + \xi^{-2}},$$

which defines $Z(p, \xi)$. For large enough $\xi$ we find that

$$Z(p, \xi) \sim \begin{cases} (ap)^{-\sigma} & p \gg \xi^{-1}, \\ (a/\xi)^{-\sigma} & p \ll \xi^{-1}. \end{cases}$$

In this case, can you infer the asymptotic forms for $G(r)$ similar to those stated above?

(2) The partition function of a scalar $D = 0$ field theory is given by

$$Z = \int dx e^{-\left(\frac{\lambda}{2} x^2 + \frac{g}{2} x^4\right)}.$$
The partition function for a Euclidean scalar field theory is defined in terms of the action $S(\phi)$ by

$$Z = \int \{d\phi\} e^{-S(\phi)/\lambda}.$$ 

$S(\phi)$ can be written explicitly in the form

$$S(\phi) = \int d^D x \left( \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \sum g_n \frac{\phi^{2n}}{2n!} + \ldots \right).$$

More interactions such as derivative interactions can be included in principle.

By inspecting the rules for generating the diagrammatic expansion of the theory show that

(i) every vertex carries a factor $\lambda^{-1}$,
(ii) every propagator line carries a factor $\lambda$.

Now consider the graphical expansion for the one-particle irreducible (1PI), truncated $n$-point function. Prove by induction or otherwise that the contribution of a given diagram carries a factor $\lambda^{L-1}$, where $L$ is the number of loops in that diagram. Hence show that the result is also true for all connected diagrams. [See Itzykson and Zuber Section 6.2 for one way to do this.]

Also show that the contribution to $W(J) = -\lambda \log Z(J)$ from diagrams with $L$ loops is $O(\lambda^L)$.

Note: in quantum field theory $\lambda$ is identified with $\bar{h}$ and so the expansion in the number of loops is the same as an expansion in powers of $\bar{h}$ which measures the size of quantum corrections.

For the $\phi^4$ field theory for $D = 4 - \epsilon$ the RG evolution equations for the coupling constant $g$ and mass $m$ are

$$\frac{du^2}{db} = 2u^2 + \frac{\Omega_D}{2(2\pi)^D} \frac{\lambda}{1 + u^2},$$

$$\frac{d\lambda}{db} = \epsilon \lambda - \frac{3\Omega_D}{2(2\pi)^D} \frac{\lambda^2}{(1 + u^2)^2},$$

where

$$u^2(b, T) = \Lambda^{-2} m^2(\Lambda, T), \quad \lambda(b, T) = \Lambda^{-\epsilon} g(\Lambda, T),$$

and $\Lambda$ is the Ultra-Violet cutoff. Note that both $u$ and $\lambda$ are dimensionless.

Verify that the non-trivial fixed point is at

$$u^* = -\epsilon/6, \quad \lambda^* = 16\pi^2 \epsilon/3.$$ 

Draw a typical trajectory flow for a theory near to $T = T_c$ for $t > 0$, $t < 0$, calculate the relevant eigenvalue $\lambda_t$ and so derive the related critical exponents.