Mathematical Tripos Part III | Michaelmas Term 2014
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Statistical Field Theory

Examples 1

Thermodynamics

(1) In standard notation, the first law of thermodynamics for a magnetic system is

$$dU = TdS - MdH .$$

where U is the internal energy, M is the magnetization and H the applied magnetic field. Define the other functions of state by appropriate Legendre transformations, for example $F = U - TS$ and hence obtain the Maxwell relations:

$$\left(\frac{\partial T}{\partial H}\right)_S = -\left(\frac{\partial M}{\partial S}\right)_H \qquad \left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H$$

$$\left(\frac{\partial S}{\partial M}\right)_T = -\left(\frac{\partial H}{\partial T}\right)_M \qquad \left(\frac{\partial T}{\partial M}\right)_S = \left(\frac{\partial H}{\partial S}\right)_M$$

(2) The specific heats at constant magnetic field and at constant magnetization for the magnetic system are

$$C_H = T\left(\frac{\partial S}{\partial T}\right)_H , \qquad C_M = T\left(\frac{\partial S}{\partial T}\right)_M .$$

The isothermal and adiabatic susceptibilities are

$$\chi_T = \left(\frac{\partial M}{\partial H}\right)_T , \qquad \chi_S = \left(\frac{\partial M}{\partial H}\right)_S .$$

Also define

$$\alpha_H = \left(\frac{\partial M}{\partial T}\right)_H .$$

From the identity

$$\left(\frac{\partial S}{\partial T}\right)_M = \left(\frac{\partial S}{\partial T}\right)_H + \left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_M ,$$

deduce that

$$\chi_T (C_H - C_M) = -T\left(\frac{\partial M}{\partial H}\right)_T \left(\frac{\partial M}{\partial T}\right)_H \left(\frac{\partial H}{\partial T}\right)_M ,$$

and hence show that

$$\chi_T (C_H - C_M) = T\alpha_H^2 \quad (\dagger) .$$

By similar means show that

$$C_H (\chi_T - \chi_S) = T\alpha_H^2 .$$

Hence show that

$$\chi_S C_H = \chi_T C_M .$$

[For this question you will need the identity

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1 . \quad]$$

(3) For $T \to T_c$ for $T < T_c$ and $H = 0$ the dependence of the following observables on T is parametrized by

$$\begin{aligned} C_H &\sim (T_c - T)^{-\alpha} \\ M &\sim (T_c - T)^{\beta} \\ \chi_T &\sim (T_c - T)^{-\gamma} . \end{aligned}$$

Using that $C_M > 0$ and (†) above derive Rushbrooke's inequality:

$$\alpha + 2\beta + \gamma \geq 2.$$

What is the equivalent set of statements for a gaseous system?

Statistical Models and Landau Theory

(4) The Hamiltonian for a set of N spins $\{\mathbf{s}_n\}$ that are 3-dimensional vectors in the presence of a magnetic field $\mathbf{H}$ is

$$\mathcal{H} = \sum_n \mu H \cos\theta_n ,$$

where θ_n is the angle between between $\mathbf{s}_n$ and $\mathbf{H}$. Show that the partition function is

$$\mathcal{Z} = \left[4\pi\left(\frac{\sinh\beta\mu H}{\beta\mu H}\right)\right]^N .$$

Compute the free energy F, the entropy S and the internal energy U. Find the equation of state and compute the susceptibility χ_T. Examine the behaviour of χ_T at low T.

(5) In a modification of the 1-dimensional Ising model the spins can take the values $\sigma_n = 1, 0, -1$. Show that the partition function is

$$\mathcal{Z} = \mathrm{Tr}\, W^n ,$$

where W is the 3×3 matrix

$$\begin{pmatrix} z\mu^2 & \mu & z^{-1} \\ \mu & 1 & \mu^{-1} \\ z^{-1} & \mu^{-1} & z\mu^{-2} \end{pmatrix}$$

with $z = e^{\beta J}$ and $\mu = e^{\beta h/2}$.

For the case $h = 0$ show that this matrix can be expressed in the form $W = P\Lambda P^{-1}$ where

$$\Lambda = \begin{pmatrix} 2\cosh\beta J & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & 2\sinh\beta J \end{pmatrix} , \quad P = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} .$$

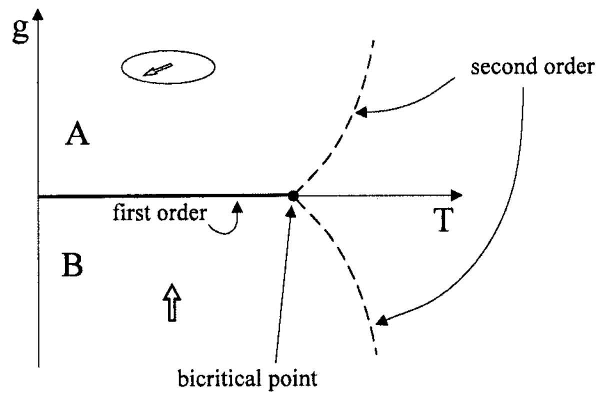
Hence find the eigenvalues of W and show that in the thermodynamic limit the free energy of the system is

$$F \;=\; -\,NkT\log\left\{\left(1+2\cosh\beta J \;+\; \sqrt{(2\cosh\beta J-1)^2+8}\right)/2\right\}\;.$$

(6) Give a plausible arguement that the phase diagram of the 3D spin model with Hamiltonian

$$H=-J\sum_{<ij>}\mathrm{s}_i\cdot\mathrm{s}_j+\frac{1}{2}g\sum_i\left((\mathrm{s}_i^z)^2-\frac{1}{2}\left((\mathrm{s}_i^x)^2+(\mathrm{s}_i^y)^2\right)\right)$$

has the form



where $-\infty<g<\infty$, $<i,j>$ means nearest neighbour pairs, and s_i is a vector at the i-th site with $|\mathrm{s}_i|=1$.

You should consider the nature of the dominant configurations for low and high temperature for $|g|$ very large, and the type of transition that is likely to separate them. Then ask what happens for low T as g changes sign. Note that in $D=3$ the $O(2)$, plane rotator, model (the spin at each site, s_i, is a unit vector lying in the xy-plane) exhibits a continuous phase transition.

[If you look in the extra material at the end of the notes on the web then you will see the answer.]

(1)

$$dU = TdS - MdH$$

Use differentiability.

(i) $\left(\frac{\partial U}{\partial S}\right)_H = T \qquad \left(\frac{\partial U}{\partial H}\right)_S = -M$

$$\left(\frac{\partial^2 U}{\partial H \partial S}\right) = \left(\frac{\partial T}{\partial H}\right)_S \qquad \left(\frac{\partial^2 U}{\partial S \partial H}\right) = -\left(\frac{\partial M}{\partial S}\right)_H$$

$$\Rightarrow \quad \left(\frac{\partial T}{\partial H}\right)_S = -\left(\frac{\partial M}{\partial S}\right)_H$$

(ii) $F = U - TS$

$$\Rightarrow \quad dF = -SdT - MdH$$

$$\Rightarrow \quad \left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H$$

(iii) $G = U + MH$

$$\Rightarrow \quad dG = TdS + HdM$$

$$\Rightarrow \quad \left(\frac{\partial H}{\partial S}\right)_M = \left(\frac{\partial T}{\partial M}\right)_S$$

$$\Phi = G + TS$$

$$\Rightarrow \quad d\Phi = -S\,dT + H\,dM$$

$$\Rightarrow \quad \left(\frac{\partial H}{\partial T}\right)_M = -\left(\frac{\partial S}{\partial M}\right)_T$$

(2) From given identity

$$C_H - C_M = -T\left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_M$$

From above have

$$\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H$$

$\Rightarrow$

$$(C_H - C_M) = -T\left(\frac{\partial M}{\partial T}\right)_H \left(\frac{\partial H}{\partial T}\right)_M$$

and also

$$\left(\frac{\partial H}{\partial T}\right)_M \left(\frac{\partial T}{\partial M}\right)_H \left(\frac{\partial M}{\partial H}\right)_T = -1$$

$\Rightarrow$ $\chi_{T_{III}}$

$$\chi_T (C_H - C_M) = -T \left(\frac{\partial M}{\partial T}\right)_H \underbrace{\left(\frac{\partial H}{\partial T}\right)_M \left(\frac{\partial M}{\partial H}\right)_T}_{-1 \, / \, \left(\frac{\partial T}{\partial M}\right)_H}$$

but also

$$\left(\frac{\partial M}{\partial T}\right)_H \left(\frac{\partial T}{\partial M}\right)_H = 1$$

$\Rightarrow$

$$\chi_T (C_H - C_M) = T \left(\frac{\partial M}{\partial T}\right)_H^2$$

or

$$\chi_T (C_H - C_M) = T \alpha_H^2$$

Have

$$\left(\frac{\partial M}{\partial H}\right)_T = \left(\frac{\partial M}{\partial H}\right)_S + \left(\frac{\partial M}{\partial S}\right)_H \left(\frac{\partial S}{\partial H}\right)_T \Rightarrow$$

$$C_H (\chi_T - \chi_S) = T \left(\frac{\partial S}{\partial T}\right)_H \left(\frac{\partial M}{\partial S}\right)_H \left(\frac{\partial S}{\partial H}\right)_T$$

$\downarrow$

Using Maxwell relation #1

$$= -T \left(\frac{\partial S}{\partial T}\right)_H \left(\frac{\partial T}{\partial H}\right)_S \left(\frac{\partial S}{\partial H}\right)_T$$

$$= T \left[\left(\frac{\partial H}{\partial S}\right)_T\right]^{-1} \left(\frac{\partial S}{\partial H}\right)_T$$

but

$$\left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial S}\right)_T = 1$$

so get

$$= T \left(\frac{\partial S}{\partial H}\right)_T^2$$

Maxwell #2

$$= T \left(\frac{\partial M}{\partial T}\right)_H^2$$

$$\Rightarrow \quad \underline{\underline{C_H (\chi_T - \chi_S) = \alpha_H^2 T}}$$

Equating both results $\Rightarrow$ $\underline{\underline{C_H \chi_S = C_M \chi_T}}$

(3) From above with $C_M > 0$ get

$$\chi_T C_H \geq T \alpha_H^2$$

$$t^{-\alpha} t^{-\gamma} \geq T\, t^{2\beta-2}$$

$$t^{-(\alpha+2\beta+\gamma-2)} \geq T \quad \text{as } t \to 0$$

$$\Rightarrow \quad \alpha+2\beta+\gamma-2 \geq 0$$

or $\quad \underline{\underline{\alpha+2\beta+\gamma \geq 2}}$

(4) Integral is over unit sphere and so

$$Z = \prod_n \left[\int d\phi_n \, d\cos\theta_n \, e^{-\beta\mu H \cos\theta_n} \right]$$

$$= \left[2\pi \frac{1}{\beta\mu H} \, 2 \sinh \beta\mu H \right]^N$$

$$\Rightarrow F = -kTN \log\left(4\pi \frac{\sinh \beta\mu H}{\beta\mu H} \right)$$

For a gaseous system:

* Specific heats are

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad C_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

* Isothermal & adiabatic compressibilities are

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T$$

Since $\rho = \frac{N}{V}$, the density.

$$K_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_S$$

* Coeff of thermal expansion is

$$\alpha_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

* Get

$$K_T(C_p - C_V) = TV\alpha_p^2$$

$$C_p(K_T - K_S) = TV\alpha_p^2$$

* So

$$K_T C_p \geq TV\alpha_p^2$$

Write

$$K_T \sim |t|^{-\gamma}$$

$$C_p \sim |t|^{-\alpha}$$

$$\alpha_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \sim |t|^{\beta-1}$$

Since $(\rho - \rho^*) \sim |t|^{\beta}$ Then

$$A\,|t|^{-\gamma-\alpha} > B\,|t|^{2\beta-2}$$

$$\Rightarrow \quad \underline{\underline{\alpha + 2\beta + \gamma \geq 2}}$$

Here $M \to (\rho - \rho^*)$ $\quad P \to h$

$$U = -\frac{\partial \log Z}{\partial \beta}$$

$$= -N \frac{\partial}{\partial \beta} \log \left(\frac{\sinh \beta \mu H}{\beta \mu H} \right)$$

$$= -N \frac{\beta \mu H}{\sinh \beta \mu H} \left[\frac{\cosh \beta \mu H}{\beta} - \frac{\sinh \beta \mu H}{\beta^2 \mu H} \right]$$

$$= -N \left[\mu H \coth \beta \mu H - \frac{1}{\beta} \right]$$

$\Rightarrow$

$$\frac{U}{N} = kT - \mu H \coth \frac{\mu H}{kT}$$

$$S = \frac{U - F}{T} \Rightarrow$$

$$\frac{S}{N} = k \left[1 - \frac{\mu H}{kT} \coth \frac{\mu H}{kT} + \log \left(4\pi \frac{\sinh (\mu H / kT)}{\mu H / kT} \right) \right]$$

$$\frac{M}{N} = -\frac{1}{N}\left(\frac{\partial F}{\partial H}\right)_T$$

$$= \mu\left[\coth\frac{\mu H}{kT} - \frac{kT}{\mu H}\right]$$

$$\coth x = \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \dots$$

$H \to 0 \quad (x \to 0)$

$$\frac{M}{N} \sim \mu\left[\frac{\mu H}{3kT} - \frac{1}{45}\left(\frac{\mu H}{kT}\right)^3 \dots\right]$$

large T have from here

$$\frac{M}{N} \sim \frac{\mu^2 H}{3kT}$$

$$\Rightarrow \chi_T \sim \frac{\mu^2}{3kT}$$

small $T \Rightarrow \frac{kT}{\mu H} \ll 1$

$$\frac{M}{N} \sim \mu \coth\frac{\mu H}{kT} - \frac{kT}{H}$$

$$U = -\frac{\partial \log Z}{\partial \beta}$$

$$= -N \frac{\partial}{\partial \beta} \log \left(\frac{\sinh \beta \mu H}{\beta \mu H} \right)$$

$$= -N \frac{\beta \mu H}{\sinh \beta \mu H} \left[\frac{\cosh \beta \mu H}{\beta} - \frac{\sinh \beta \mu H}{\beta^2 \mu H} \right]$$

$$= -N \left[\mu H \coth \beta \mu H - \frac{1}{\beta} \right]$$

$\Rightarrow$

$$\frac{U}{N} = kT - \mu H \coth \frac{\mu H}{kT}$$

$$S = \frac{U - F}{T} \Rightarrow$$

$$\frac{S}{N} = k \left[1 - \frac{\mu H}{kT} \coth \frac{\mu H}{kT} + \log \left(4\pi \frac{\sinh (\mu H / kT)}{\mu H / kT} \right) \right]$$

$$\frac{M}{N} = -\frac{1}{N}\left(\frac{\partial F}{\partial H}\right)_T$$

$$= \mu\left[\operatorname{Coth}\frac{\mu H}{kT} - \frac{kT}{\mu H}\right]$$

$$\operatorname{Coth} x = \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \cdots$$

$H \to 0 \quad (x \to 0)$

$$\frac{M}{N} \sim \mu\left[\frac{\mu H}{3kT} - \frac{1}{45}\left(\frac{\mu H}{kT}\right)^3 \cdots\right]$$

large T have from here

$$\frac{M}{N} \sim \frac{\mu^2 H}{3kT}$$

$$\Rightarrow \chi_T \sim \frac{\mu^2}{3kT}$$

Small T $\Rightarrow \frac{kT}{\mu H} \ll 1$

$$\frac{M}{N} \sim \mu \operatorname{Coth}\frac{\mu H}{kT} - \frac{kT}{H}$$

$$\Rightarrow \quad \chi_T \sim \frac{\mu^2/kT}{\sinh^2 \mu H/kT} + \frac{kT}{H^2}$$

$\Rightarrow$

$$\chi_T \sim 4\frac{\mu^2}{kT} e^{-2\mu H/kT} + \frac{kT}{H^2}$$

Small in this limit

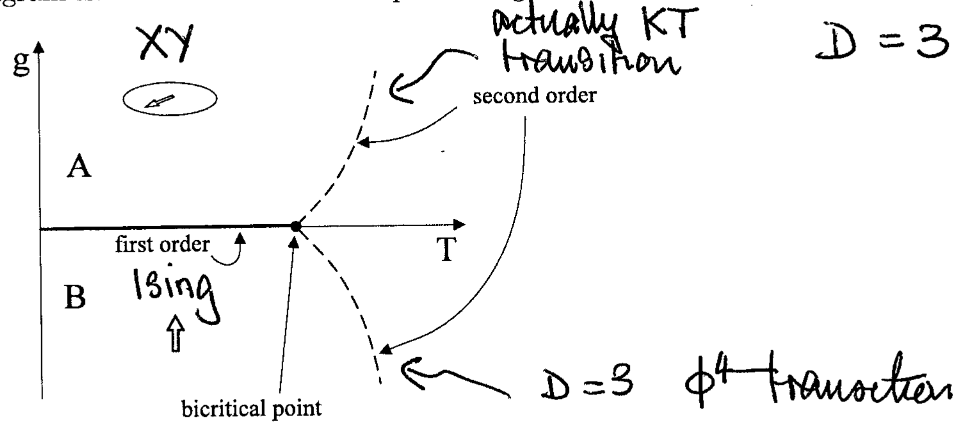
(5)

$$W_{\sigma\sigma'} = e^{\beta J \sigma\sigma'} \cdot e^{-\frac{1}{2}\beta h(\sigma+\sigma')}$$

	1	0	−1
1	$e^{\beta J} e^{\beta h}$	$e^{\frac{1}{2}\beta h}$	$e^{-\beta J}$
0	$e^{\frac{1}{2}\beta h}$	1	$e^{-\frac{1}{2}\beta h}$
−1	$e^{-\beta J}$	$e^{-\frac{1}{2}\beta h}$	$e^{\beta J} e^{-\beta h}$

$z = e^{\beta J} \quad \mu = e^{\frac{1}{2}\beta h}$

A **bicritical point** is a critical point at which two critical lines terminate. A typical phase diagram is: A model which has a phase diagram like this is given by



the Hamiltonian

$$H = -J \sum_{<ij>} \mathbf{s}_i \cdot \mathbf{s}_j + \frac{1}{2} g \sum_i \left((\mathbf{s}_i^z)^2 - \frac{1}{2}((\mathbf{s}_i^x)^2 + (\mathbf{s}_i^y)^2) \right)$$

$< ij >$ means nearest neighbour pairs, i.e., it labels the links on the lattice.

$\mathbf{s}_i$ is a vector at the i-th site with $|\mathbf{s}_i| = 1$.

★ For **low** T thermal fluctuations can be ignored and it is safe to just find the configuration (i.e., set of values) of spins $\{\mathbf{s}_i\}$ which minimizes H. Since $J > 0$ the first term causes the spins to align with each other to give ferromagnetic ordering.

(i) $g < 0$. Ordering is preferred along z-axis. This is phase B.

(ii) $g > 0$. Ordering is preferred in the xy-plane $\perp$ z-axis. This is phase A.

(iii) $g = 0$. Neither A nor B preferred: two-phase coexistence – it is a first-order line and the transition occurs as g changes sign at low T..

★ For high T ordering is absent: it is destroyed by thermal fluctuations. Consider $|g|$ large and increase T from low to high value.

(i) $g < 0$: In this regime the fluctuations in the xy-plane are negligible and we may set $s_i^x, s_i^y \approx 0$ and only the z-component survives and we have an Ising model. As T increases we then observe the second-order phase transition of the 3D Ising model. This is the dotted line for $g < 0$.

(ii) $g > 0$: In this regime the fluctuations in s_i^z are negligible and we recover the $O(2)$, plane rotator, model which exhibits second-order behaviour as T increases. this gives the i dotted line for $g > 0$.

All lines cannot terminate abruptly since the LG theory predicts that the change between different potential shapes is continuous. Therefore, in the absence of any more structure, the lines of transitions must join up as shown.

★ Another thing to note is that the surfaces to the low-T side of the lines of critical points are first-order surfaces. This can be seen by imposing a magnetic field **h** on the system with components both $\|$ and $\perp$ to the z-axis and adding the magnitude, h, of **h** (including sign) as a third orthogonal axis to generate a 3D phase plot of which our 2D plot is the $h = 0$ cross-section. Then as h changes sign the magnetisation, **M**, changes discontinuously at $h = 0$. This occurs as the surfaces in our 2D phase diagram on the low-T side of the critical lines are punctured, and hence they are in fact first-order surfaces. Of course, because the order parameter is a vector the possible patterns of behaviour and the competition between the effects of the terms governed by the coupling, g, and by h is, in general, complicated.

In this case, the line $g = 0$ at low T is revealed as a line of multiphase coexistence.

* $dU = TdS - MdH$

$$d\hat{U} = TdS + HdM$$

Just alternative definitions of internal energy. All thermodynamic consequences the same

- $U = U(S,H)$ $\hat{U} = \hat{U}(S, M)$ are related by Legendre transformation
- H is better independent variable than $M \Rightarrow$ it is a coupling / chemical potential

* H is due to ~~external~~ currents in solenoid etc : $\underline{\nabla} \wedge \underline{H} = \underline{j}$

* Fixed current I $(\equiv \underline{j}) \Rightarrow \underline{H}$ fixed

* $\underline{B} = \mu_0 \underline{H} + \underline{M}$

* $\delta E = \underline{H} \cdot \delta \underline{B}$

$$= \delta\left(\frac{1}{2}\mu_0 \underline{H}^2\right) + \underline{H} \cdot \delta \underline{M}$$

Thus, for fixed $\underline{H} \Rightarrow$ fixed $\underline{I}$ we assign

$$\delta W = \underline{H} \cdot \delta \underline{M}$$

This corresponds to $\hat{U}$ above and is derived by considering the total energy of the sample and magnetic field in the environment.

* The system is sample + environment field.

* Consider fixed $\underline{M}$ i.e. a distribution of magnetic dipoles that remains unchanged through the process.

$$E = -\underline{M}\cdot\underline{H}$$

$$\Rightarrow \quad \delta W = -\underline{M}\cdot\delta\underline{H}$$

which is consistent with the definition of U. Note that the energy of the magnetic field in the environment is not included in the variation.

* Work done on surface currents with $\underline{J}_s = \underline{M}$ has (l = length sample)

$$\delta W = l\,\underline{J}_s\cdot\underline{E}\,dt = -l\,\underline{M}\,\underbrace{A\,\delta\underline{H}}_{\delta\phi}$$

$\delta\phi \leftarrow \phi$ is flux

$$\text{or} \quad \delta W = -V\,\underline{M}\cdot\delta\underline{H}$$

Again, only the sample included

Note if allow $\underline{M}$ to vary because population of spin d.o.f. changes then from $E = -\underline{M} \cdot \underline{H}$ have

$$\delta W = -\delta \underline{M} \cdot \underline{H} - \underline{M} \cdot \delta \underline{H}$$

heat   work

It is heat because it is due to population change. Thus, it all depends what you isolate as heat and as work.

①

$$(C_H - C_M) = -T\left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_M$$

Now

$$\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H$$

$$(C_H - C_M) = -T\left(\frac{\partial M}{\partial T}\right)_H \left(\frac{\partial H}{\partial T}\right)_M$$

$$\left(\frac{\partial H}{\partial T}\right)_M \cdot \left(\frac{\partial T}{\partial M}\right)_H \left(\frac{\partial M}{\partial H}\right)_T = -1$$

$$\chi_T (C_H - C_M) = -T \underbrace{\left(\frac{\partial M}{\partial T}\right)_H}_{\alpha_H} \underbrace{\left(\frac{\partial H}{\partial T}\right)_M \left(\frac{\partial M}{\partial H}\right)_T}_{-1 / \left(\frac{\partial T}{\partial M}\right)_H}$$

but

$$\left(\frac{\partial M}{\partial T}\right)_H \left(\frac{\partial T}{\partial M}\right)_H = 1.$$

$\Rightarrow$

$$\chi_T (C_H - C_M) = T\left(\frac{\partial M}{\partial T}\right)_H^2$$

$$\underline{\underline{\chi_T (C_H - C_M)}} = T\alpha_H^2$$

$$\left(\frac{\partial M}{\partial H}\right)_T = \left(\frac{\partial M}{\partial H}\right)_S + \left(\frac{\partial M}{\partial S}\right)_H \left(\frac{\partial S}{\partial H}\right)_T$$

$$C_H \left(\chi_T - \chi_S\right) = T \left(\frac{\partial S}{\partial T}\right)_H \left(\frac{\partial M}{\partial S}\right)_H \left(\frac{\partial S}{\partial H}\right)_T$$

$=$ Maxwell (1)

$$= -T \left(\frac{\partial S}{\partial T}\right)_H \left(\frac{\partial T}{\partial H}\right)_S \left(\frac{\partial S}{\partial H}\right)_T$$

$$\left(\frac{\partial S}{\partial T}\right)_H \left(\frac{\partial T}{\partial H}\right)_S \left(\frac{\partial H}{\partial S}\right)_T = -1$$

$$= -T \left[\left(\frac{\partial H}{\partial S}\right)_T\right]^{-1} \left(\frac{\partial S}{\partial H}\right)_T$$

but

$$\left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial S}\right)_T = 1$$

$$= T \left(\frac{\partial S}{\partial H}\right)_T^2$$

Maxwell (2).

$$C_H(\chi_T - \chi_S) = T\left(\frac{\partial M}{\partial T}\right)_H^2$$

$$= \alpha_H^2 T$$

$\Rightarrow$

$$C_H \cancel{\chi_T} - C_H \chi_S = \cancel{\chi_T C_H} - \chi_T C_M$$

$$\Rightarrow \quad C_H \chi_S = C_M \chi_T$$

$$\chi_T C_H \geq T \alpha_H^2$$

$$t^{-\alpha} t^{-\gamma} \geq T t^{2\beta - 2}$$

$$t^{-(\alpha + \beta + \gamma - 2)} \geq T \quad \text{as } t \to 0$$

$$\Rightarrow \quad \alpha + 2\beta + \gamma - 2 \geq 0$$

i.e. $\alpha + 2\beta + \gamma \geq 2$

$* \quad V = N\rho_c(T) = N\rho^*(T) + N\rho_c$

constant

So $\alpha_p \simeq \left(\frac{\partial V}{\partial T}\right)_p \sim \frac{\partial \rho^*}{\partial t} \sim |t|^{\beta-1}$

since $\rho^*(t) \sim |t|^{\beta}$

$* \; K_T \sim t^{-\gamma}$

$* \quad C_p \sim t^{-\alpha}$

$C_p K_T > \text{const} \cdot \alpha_p^2$

$t^{-\alpha} t^{-\gamma} > \text{const} \; t^{(2\beta - 2)}$

$t^{-(\alpha + 2\beta + \gamma - 2)} > \text{Const}$

$\Rightarrow \quad \underline{\underline{\alpha + 2\beta + \gamma \geq 2}}$