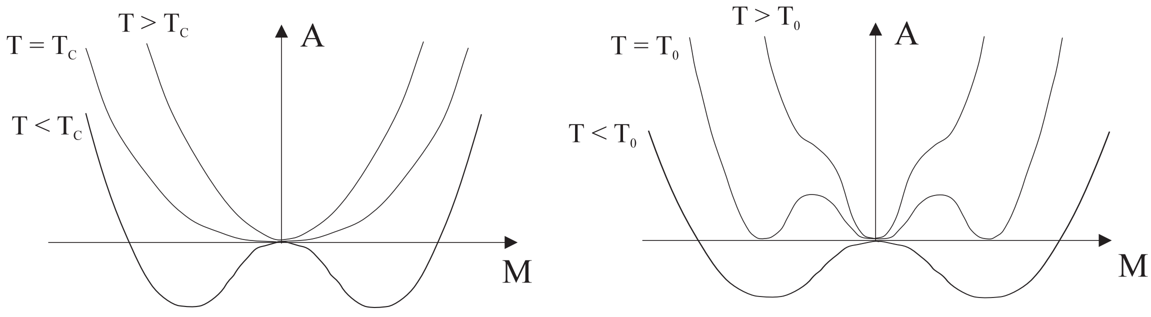
$\mathbf{A_4 \leq 0}$

As T decreases $A(M)$ behaves qualitatively differently depending on whether $A_4 > 0$ or $A_4 \leq 0$:



second-order transition at $T = T_C$

first-order phase transition at T_0 , $T_0 > T_C$

Hence the system passes from a second-order transition to a first-order transition as A_4 changes sign and becomes negative.

The stationary points are at $M = 0$ and at

$$M^2 = \left[-A_4 \pm \left(A_4^2 - 4A_2A_6 \right)^{\frac{1}{2}} \right] / 2A_6 \equiv M_\pm^2 .$$

The $+$ sign gives the minima and the $-$ sign the maxima.

T_0 is determined by $A(M) = 0$ having a double root at $M = \pm M_+$ (note that A_0 is set to zero so that $A(0) = 0$ is the minimum for $T > T_0$). The solution is

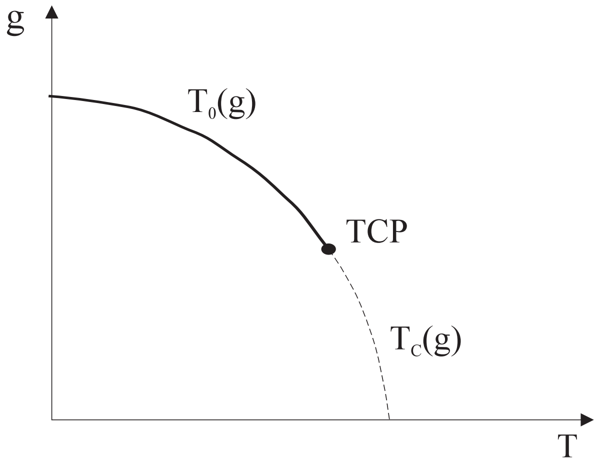
$$A_2 = \frac{3}{16} \frac{A_4^2}{A_6}, \tag{1}$$

and at the transition

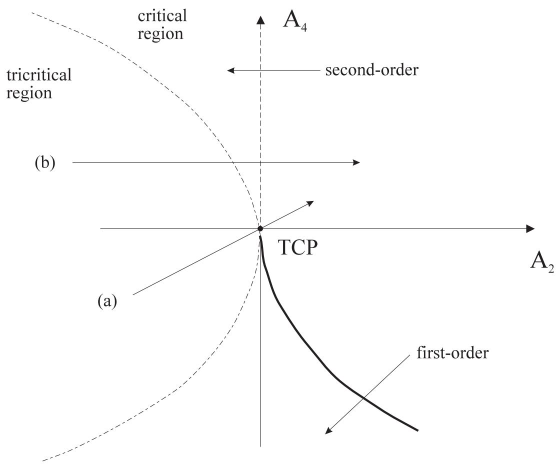
$$\left(\Delta M^2 \right)^2 = M_+^2 = -\frac{3}{4} \frac{A_4}{A_6} \tag{2}$$

Thus the point $T = T_c, A_4 = 0$ separates the first-order line from the second-order line: this is a **tricritical point**. To see the tricritical point these two parameters have to take these special values and this requires tuning two external fields in the phase diagram.

In the space of physical fields, denoted by T and g (e.g., g can be identified with a chemical potential controlling the relative abundances in a two component system), the phase diagram has the form:



Alternatively the phase diagram in (A_2, A_4) space takes the form:



(i) **Both** trajectories (a) and (b) exhibit a second-order transition.

(ii) Trajectory (a) passes through the TCP and lies entirely within the tricritical region. The transition is characterised entirely by the properties of the TCP, and all critical exponents are tricritical ones.

(iii) Trajectory (b) exhibits an **ordinary** second order transition. However, it starts in the tricritical region and so initially the divergence of the relevant quantities is controlled by the TCP. Eventually it passes into the critical region and the transition is characterised by the line of ordinary critical points and the critical exponents that are given above.

In other words we only see a transition controlled by the TCP when we apporoach along a trajectory lying in the tricritical region. For trajectories that pass from one region to another we see a change in the critical behaviour. This change is characterised by **crossover exponents**.

At the TCP $A_4 = 0$, and so we have

$$|M| = \left|\frac{A_2}{A_6}\right|^{\beta_t} \quad \text{with} \quad \beta_t = \frac{1}{4}.$$

New exponents can be defined, see Eqns (1),(2):

$$\begin{aligned}(T_0 - T_c) &\sim (-A_4)^{\frac{1}{\psi}} \quad \text{with} \quad \psi = \frac{1}{2}\\ \Delta M &\sim |A_4|^{\beta_u} \quad \text{with} \quad \beta_u = \frac{1}{2},\end{aligned}$$

where ΔM is the discontinuity in M across the first-order line ($A_4 < 0$). then for small A_4 we can write

$$\begin{aligned}|M| &= \left|\frac{A_2}{A_6}\right|^{\beta_t} m(x),\\ x &= \frac{A_4}{2|A_2|^{\frac{1}{2}} A_6^{\frac{1}{2}}}\end{aligned}$$

where x is a dimensionless variable which vanishes as $A_4 \to 0$. Note that A_2 remains **non-zero** at a TCP. To see tricritical behaviour along a trajectory we clearly need x small, i.e.,

$$A_4^2 \ll 4|A_2|A_6.$$

This defines a parabola in the (A_4, A_6) plane separating the tricritical from critical regions. This is shown on the figure. In the space of physical parameters it translates into a similar shaped curve defining the two regions controlled by the TCP and ordinary critical points respectively.

The general theory of continuous phase transitions can be encoded in terms of scaling functions and relies on dimensional analysis together with some assumptions about the behaviour of the scaling functions for small argument. If naive or engineering dimensions are used this is generally a recoding of Landau theory but is often used to describe the behaviour of the relevant thermodynamic variables as a function of the actual external fields and hence parametrises the experimental observations.
Add a magnetic field, h, with contribution to the free energy of $-hM$. Then we can always write

$$A = \frac{|A_2|^{\frac{3}{2}}}{A_6^{\frac{1}{2}}} F\left(\frac{A_4}{2|A_2|^{\frac{1}{2}} A_6^{\frac{1}{2}}}, \frac{hA_6^{\frac{1}{4}}}{|A_2|^{\frac{5}{4}}}\right).$$

The point is that the **equilibrium** free energy, A, can always be written in terms of dimensionless ratios in this way. As before assign dimension (-1) to M and dimension d to A, and then A_n has dimension $(d+n)$. The above expression is then a general way of writing the dependence of A at equilibrium on the coefficients

A_{2n} in terms of a **scaling function**, F. Note that since A_6 is always taken as positive it causes no problem in the denominators.
We now compare with a standard form parametrising A near the TCP:

$$A = |T - T_c(\tilde{g})|^{2-\alpha}\ F\left(\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}}, \frac{h}{|T - T_c(\tilde{g})|^{\Delta}}\right),$$

where $\tilde{g} \propto A_4$, and thus measures the distance from the TCP along the tangent to the critical line at the TCP. Note that $\tilde{g}$ has been substituted for the field g as the second independent external field: the critical line is thus parametrised as $T_c(\tilde{g})$. The TCP then is at position $(0, T_t)$ in the $(\tilde{g}, T)$ plane where $T_t = T_c(0)$. Labelling the critical exponents at the TCP by suffix, t, we clearly have

$$\alpha_t = \frac{1}{2}, \qquad \phi_t = \frac{1}{2}, \qquad \Delta_t = \frac{5}{4}.$$

The following examples clarify the interpretation.

(i) $h = 0,\ \tilde{g} \to 0,\ T \to T_t$ such that $\frac{\tilde{g}}{|T-T_t|^{\phi_t}} \equiv x$ is fixed. Then

$$A = |T - T_t|^{\frac{3}{2}} F(x, 0) \qquad \text{with} \qquad F(0,0) \quad \textbf{finite}.$$

We see tricritical behaviour and since $\tilde{g} \sim |T - T_t|^{\phi_t}$ the trajectory lies in the tricritical region. ϕ_t is the **cross-over exponent**.

(ii) $h \to 0,\ \tilde{g}$ fixed, $T \to T_c$.

$$A = |T - T_c|^{\frac{3}{2}}\ F\left(\frac{\tilde{g}}{|T - T_c|^{\frac{1}{2}}}, 0\right).$$

the argument of F is not under control and so we rearrange the expression:

$$A = \frac{|T - T_c|^2}{\tilde{g}}\ G\left(\frac{|T - T_c|^{\frac{1}{2}}}{\tilde{g}}, 0\right),$$

where $G(z, 0) = zF(\frac{1}{z}, 0)$ and $G(0, 0)$ is finite and non-zero. This property of G is an assumption in the general theory and could be violated. It **does** follow from the standard Landau analysis and hence if it turned out to be false in an experiment it would signal a breakdown of the Landau theory. The goal then would be to rescue the dimensional analysis approach by assigning values to the dimensions of the parameters different from the naive ones but which render the scaling functions F and G well behaved for small argument.

In this case we find that A shows the normal critical behaviour associated with an ordinary critical point, namely

$$A \sim |T - T_c|^{2-\alpha} \qquad \text{with} \qquad \alpha = 0.$$