Natural Sciences Tripos Part IA
Mathematics III (B course)
Easter 2006

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May 22, 2006

Contents

2 Partial Differential Equations

2.1 Introduction

In science we wish to investigate the behaviour of functions of more than one variable. E.g.,

- the vibrating string where the displacement at position x and time t is $y(x,t)$.
- The amount of a substance at x and t diffusing in a medium measured by its concentration $\Theta(x,t)$. E.g., ink drop in water.
- The electrostatic potential $\phi(x,y)$ due to a distribution of charge with charge density $\rho(x,y)$.

Each of these functions satisfies a **partial differential equation** (PDE) characteristic of the physical phenomenon being studied. A PDE is an equation relating a function $f(x,y,\ldots)$ of **more than one variable** and its partial derivatives with respect to $x, y, \ldots$. Define notation

$$f_x = \frac{\partial f}{\partial x}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad \text{etc.,}$$

where each suffix denotes partial differentiation with respect to (w.r.t) that variable. The PDE is then of the form

$$\mathcal{F}(x, y, \ldots, f_x, f_y, \ldots, f_{xx}, f_{xy}, f_{yy}, \ldots) \ = \ 0.$$

- The **order** of the PDE is the order of the highest derivative appearing in $\mathcal{F}$.
- The PDE is **linear** if $\mathcal{F}$ contains no powers of $f, f_x, \ldots$ higher than 1.

To define a unique solution to an ordinary differential equation (ODE) we need extra pieces of information in form of the values of f and/or its derivatives at a number of points to fix up the arbitrary constants of integration. These are given by the particular conditions of the system being studied.

To obtain a unique solution to a PDE we need extra information in the form of values of f etc., on surfaces in $(x, y, \ldots)$ space. Usually, a solution is sought for $(x, y, \ldots)$ in some region $\mathcal{D}$ and the extra information or **boundary conditions** are given on all or part of the **boundary** $\partial\mathcal{D}$ of this region. They fix up arbitrary functions arising in the integration. E.g.,

$$\begin{aligned} \sigma(x,y) &= y^2 + g(x)\ , \\ \frac{\partial\sigma}{\partial y} &= 2y, \quad \text{independent of } \ g(x). \end{aligned}$$

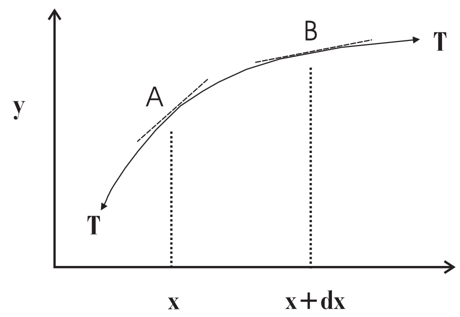
$$\begin{aligned} \sigma(x,y) &= yf(x)\ , \\ y\frac{\partial\sigma}{\partial y} &= \sigma, \quad \text{independent of } \ f(x). \end{aligned}$$

To work out how much information is sufficient but not too much is generally a hard problem.

In these lectures we shall study examples of **linear PDEs of second order** and we shall study them in the context of their physical application.

2.2 Physical derivation of important equations

2.2.1 The wave equation



Look at **vibrating string**. Let the string be under tension T with mass per unit length ρ. The displacement at position x and time t is $y(x,t)$.

Consider element AB of length dx.

- The transverse force F obtained by resolving in y-direction is

$$\begin{aligned} F &= \approx T(\text{slope at B } - \text{ slope at A}) \\ &= T\left(\frac{\partial y}{\partial x}(x+dx,t) - \frac{\partial y}{\partial x}(x,t)\right) \ \approx \ T\frac{\partial^2 y}{\partial x^2}dx\ . \end{aligned}$$

- Mass of element AB is ρdx.

Then Newton's Law is

$$T\frac{\partial^2 y}{\partial x^2}dx = \rho dx\frac{\partial^2 y}{\partial t^2}.$$

We then get the **wave equation** for wave motion in the string

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}\,, \qquad c = \sqrt{T/\rho}\,. \qquad \text{Wave equation}}$$

We shall see that c is the velocity of the waves.

For problems in higher dimensions, such as sound waves, vibrations of drum membrane etc., the equation becomes

$$\nabla^2\psi(\boldsymbol{x},t) = \frac{1}{c^2}\frac{\partial^2\psi(\boldsymbol{x},t)}{\partial t^2}\,,$$

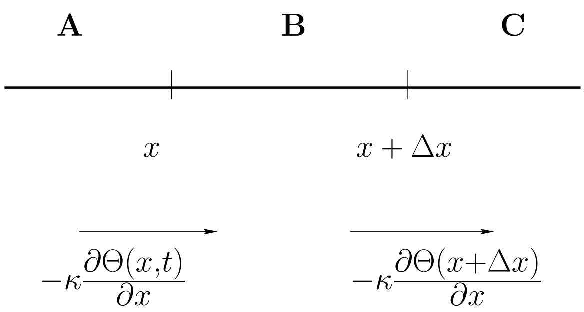
where $\psi(\boldsymbol{x},t)$ is the displacement from equilibrium.

Just as in solving for particle motion the boundary conditions are the initial conditions for the position and velocity of **each segment** of the string. We need

$$\begin{aligned} y(x,0) &= y_0(x)\,, \qquad \text{initial position,} \\ \frac{\partial y}{\partial t}(x,0) \equiv y_t(x,0) &= v_0(x)\,, \qquad \text{initial velocity}\,. \end{aligned}$$

2.2.2 The heat or diffusion equation

Consider a substance diffusing in 1D and let the concentration (i.e., density) be $\Theta(x,t)$ at position x and time t.



The rate of diffusion from A to B is proportional to the concentration gradient.

$$R(A\to B) = -\kappa\frac{\partial\Theta(x,t)}{\partial x}, \qquad R(B\to C) = -\kappa\frac{\partial\Theta(x+\Delta x,t)}{\partial x}\,,$$

Here κ is the constant of diffusion.

Hence the rate of change of substance in region B is

$$\begin{aligned}\frac{\partial[\Theta(x,t)\Delta x]}{\partial t} &= R(A\to B) - R(B\to C)\\ &= \kappa\left[\frac{\partial\Theta(x+\Delta x,t)}{\partial x} - \frac{\partial\Theta(x,t)}{\partial x}\right] .\end{aligned}$$

OR

$$\boxed{\frac{\partial\Theta(x,t)}{\partial t} = \kappa\frac{\partial^2\Theta(x,t)}{\partial x^2} . \qquad \text{Heat/Diffusion equation}}$$

Drop a drop of ink into water and it spreads out as a cloud with density $\Theta(\boldsymbol{x},t)$ at time t. $\Theta(\boldsymbol{x},t)$ obeys the **Diffusion equation**

$$\frac{\partial\Theta(\boldsymbol{x},t)}{\partial t} = \kappa\nabla^2\Theta(\boldsymbol{x},t) .$$

A closely related equation is the Schrödinger equation for a free particle in quantum mechanics:

$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{x},t) .$$

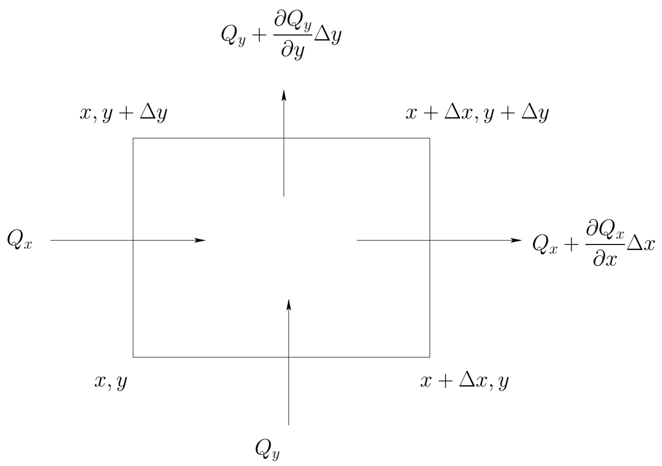
$\hbar$ is Planck's constant and m is the particle mass. The analogy with the heat equation is of very great importance.

The boundary condition in this case is to give the initial value of Θ:

$$\Theta(x,0) = \Theta_0(x) .$$

This is physically sensible since the spot of ink has an initial shape which is all we need to know to predict how it will spread.

2.2.3 Laplace's equation



Consider the temperature distribution $\phi(x, y)$ in a 2D body in equilibrium: ϕ is independent of time t.

Q_x (Q_y) is the heat-energy flux in the x (y) direction.

The phenomenological physical law giving Q_x and Q_y is

$$\begin{aligned} Q_x &= -\sigma \frac{\partial \phi(x,y)}{\partial x} \Delta y \,, \\ Q_y &= -\sigma \frac{\partial \phi(x,y)}{\partial y} \Delta x \,. \end{aligned}$$

I.e., Q is proportional to minus the temperature gradient and the length of the edge. The constant of proportionality is the thermal conductivity σ.

The **total** influx of heat-energy is **zero**. Thus

$$\left[\left(Q_x + \frac{\partial Q_x}{\partial x} \Delta x \right) - Q_x \right] \; + \; \left[\left(Q_y + \frac{\partial Q_y}{\partial y} \Delta y \right) - Q_y \right] \;=\; 0 \,,$$

$\Longrightarrow$

$$\frac{\partial Q_x}{\partial x} \Delta x + \frac{\partial Q_y}{\partial y} \Delta y \;=\; 0$$

OR

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) (-\sigma \Delta x \Delta y) \;=\; 0 \,.$$

Thus

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \;=\; 0 \,. \qquad \text{Laplace's Equation}}$$

Another application is to the electrostatic potential $\phi(\boldsymbol{x})$ due to a distribution of charge density $\rho(\boldsymbol{x})$. The electric field satisfies

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) \;=\; \frac{1}{\varepsilon_0} \rho(\boldsymbol{x}), \quad \text{and} \quad \boldsymbol{E}(\boldsymbol{x}) \;=\; -\nabla \phi(\boldsymbol{x}) \,.$$

Hence, get

$$\nabla^2 \phi(\boldsymbol{x}) \;=\; -\frac{1}{\varepsilon_0} \rho(\boldsymbol{x}) \,, \qquad \text{Poisson's equation.}$$

This reduces to Laplace's equation when $\rho(\boldsymbol{x}) = 0$.

The boundary conditions are more varied here. They typically take one of two forms giving data on the whole boundary $\partial \mathcal{D}$ to the region $\mathcal{D}$ where the solution is needed. The two main choices are

i) **Dirichlet condition**

Give the **value** of ϕ on ∂D. This would be typical if we know the temperature on the boundary and want to know it in the interior.

$$\phi(\boldsymbol{x}) \;=\; \phi_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\mathcal{D} \;.$$

(ii) **Neumann condition**

Give the **normal derivative** of ϕ on $\partial\mathcal{D}$. Typical of electrostatic problems where we are given the electric field $\boldsymbol{E}$ at the boundary and wish to calculate the potential ϕ inside.

$$\boldsymbol{n}(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x}) \equiv \frac{\partial\phi}{\partial n}(\boldsymbol{x}) \;=\; \phi_n(\boldsymbol{x}) \qquad \begin{cases} \boldsymbol{x} \in \partial\mathcal{D}, \\ \boldsymbol{n}(\boldsymbol{x}) \text{ is unit surface normal at } \boldsymbol{x} \end{cases} .$$

Note that $\boldsymbol{n}(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x}) = -\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{E}(\boldsymbol{x})$.

It is possible to generalize and give a linear combination $\alpha\phi(\boldsymbol{x}) + \beta\dfrac{\partial\phi}{\partial n}$ on $\partial\mathcal{D}$.

2.3 Classification

This is a statement of terminology.

Consider the general form in 2D

$$a\frac{\partial^2\psi}{\partial x^2} + 2b\frac{\partial^2\psi}{\partial x\partial y} + c\frac{\partial^2\psi}{\partial y^2} + f\frac{\partial\psi}{\partial x} + g\frac{\partial\psi}{\partial y} + h\psi \;=\; 0,$$

where a, b, etc. are constants.

(i) **Elliptic**

The equation is **elliptic** if $b^2 < ac$. Example is Laplace's equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \;=\; 0 \;\;.$$

(ii) **Parabolic**

The equation is **parabolic** if $b^2 = ac$. Example is the Heat equation

$$\kappa\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial t} \;=\; 0 \;\;.$$

(iii) **Hyperbolic**

The equation is **hyperbolic** if $b^2 > ac$. Example is the Wave equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial x^2} \;=\; 0 \;\;.$$

2.4 Methods of solution

2.4.1 Method for some elliptic and hyperbolic equations in 2D

Consider equations of the form

$$a\frac{\partial^2\psi}{\partial x^2} + 2b\frac{\partial^2\psi}{\partial x\partial y} + c\frac{\partial^2\psi}{\partial y^2} \;=\; 0.$$

Look for a solution of the form

$$\psi(x,y) \;=\; f(x+py) \;\equiv\; f(z),$$

where p is a constant and $z = x + py$.

Then use chain rule:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz},\\ \frac{\partial f}{\partial y} &= \frac{df}{dz}\frac{\partial z}{\partial y} = p\frac{df}{dz}.\end{aligned}$$

Substitute into the equation to get

$$a\frac{d^2 f}{dz^2} + 2bp\frac{d^2 f}{dz^2} + cp^2\frac{d^2 f}{dz^2} \;=\; 0.$$

Hence we find

$$cp^2 + 2bp + a \;=\; 0,$$

with roots $p = p_+, p_-$ given by

$$p_+ = -b + \sqrt{b^2 - ac}, \qquad p_- = -b - \sqrt{b^2 - ac}.$$

These roots will be **complex** if $b^2 < ac$ i.e., for **elliptic** equations. In this case we see that $p_+ = p_-^*$.

Let $u = x + p_+ y,\ v = x + p_- y$ and then, because the equation is **linear** we can take a **linear combination** of independent solutions to be the general solution. So we find

$$\psi(x,y) \;=\; f(x+p_+y) + g(x+p_-y) \;=\; f(u) + g(v),$$

where f and g are arbitrary functions of a single variable. These are the analogues of the arbitrary constants in ordinary differential equations.

Two special cases are:

(a) The wave equation
$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2},$$
with $u = x - ct,\ v = x + ct$ and solution of the form
$$\psi(x,t) = f(x-ct) + g(x+ct)\ .$$
An important example of this kind of solution is
$$\psi(x,t) = \sin[k(x-ct)]\ ,$$
with k an arbitrary constant. This is a wave travelling at velocity c. More on this shortly.

(b) Laplace's equation
$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0,$$
With $u = x+iy,\ v = x-iy$. Then we can write the general form most neatly as
$$\psi(x,y) = f(x+iy) + g(x-iy) \equiv f(z) + g(z^*) \ \text{ where } \ z = x+iy\ .$$
It may seem odd that the solution becomes complex when we started out with ψ real. However, because the equation is **linear** with real coefficients it must be that the real and imaginary parts of this solution **separately** satisfy the equation. Thus for our purposes we can restrict our solution to
$$\psi(x,y) = \text{Real}[f(z)+g(z^*)]\ , \quad \text{or} \quad \psi(x,y) = \text{Imag}[f(z)+g(z^*)]\ .$$
Examples are:
$$\begin{aligned}\psi(x,y) &= \text{Real}(z) &&= x\ , && \text{trivially satisfies the equation}\\ &= \text{Real}(z^3) &&= x^3 - 3xy^2\ , &&\\ &= \text{Real}(e^{inz}) &&= e^{-ny}\cos(nx)\ , && n \text{ an integer constant}\end{aligned}$$

2.4.2 Separation of variables

Suppose $b = 0$. The general equation is now
$$a\frac{\partial^2\psi}{\partial x^2} + c\frac{\partial^2\psi}{\partial y^2} + f\frac{\partial\psi}{\partial x} + g\frac{\partial\psi}{\partial y} + h\psi = 0.$$
Try a solution of the **separable** form $\psi(x,y) = X(x)Y(y)$. This will **not** be the most general solution since this would not be separable in this way. However, we will see that this is a very useful move.

We then get

$$aY\frac{d^2X}{dx^2} + cX\frac{d^2Y}{dy^2} + fY\frac{dX}{dx} + gX\frac{dY}{dy} + hXY \ = \ 0 \ .$$

OR, dividing by XY and rearranging slightly

$$\frac{1}{X}\left[a\frac{d^2X}{dx^2} + f\frac{dX}{dx} + hX\right] \ = \ -\frac{1}{Y}\left[c\frac{d^2Y}{dy^2} + g\frac{dY}{dy}\right] \ = \ \lambda \quad \text{a constant} \ .$$

This must be true since the first term depends only on x and the second only on y and hence they cannot be equal unless they are **independent** of both x, y.

Get two ODEs which we solve using standard methods:

$$\begin{aligned} a\frac{d^2X}{dx^2} + f\frac{dX}{dx} + (h-\lambda)X &= 0 \ , \\ c\frac{d^2Y}{dy^2} + g\frac{dY}{dy} + \lambda Y &= 0 \ . \end{aligned}$$

This is as far as we can go without knowing the specific problem under study and its boundary conditions.

It turns out that not necessarily all values of λ are allowed. Those that are are will be determined by the boundary conditions and might be a discrete set. We shall see examples soon, but some are the allowed frequencies of a plucked string and the values of the allowed energy levels in an atom. They are actually examples of **eigenvalues**.

For each **allowed** value of λ we label the separable solution with λ:

$$\psi_\lambda(x,y) \ = \ X_\lambda(x)Y_\lambda(y) \ .$$

Also, because the equations are linear, then $\alpha\psi_\lambda$ is also a solution, α a constant. We then choose the **normalization** of ψ_λ by some convenient procedure to make life easiest. The general solution is the linear combination

$$\psi(x,y) \ = \ \sum_\lambda \alpha_\lambda \psi_\lambda(x,y) \ ,$$

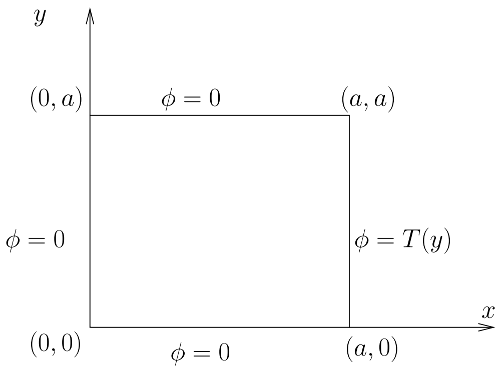
where the sum is over the allowed values of λ and the α_λ are constants.

[Note: if $b \neq 0$ in the equation and so there is a $\partial^2\psi/\partial x\partial y$ term then a change of variables to $w = x + \alpha y, z = x + \beta y$ will give an equation of suitable form (with $b = 0$) for right choice of constants α, β.]

2.5 Laplace's equation

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0$$

Q What is the steady-state temperature distribution $\phi(x,y)$ for a region bounded by a square whose boundary, or edges, are maintained at the temperatures shown in the figure? Namely, three edges at $\phi = 0$ and the fourth at $\phi(a,y) = T(y)$. These are Dirichlet conditions. The square has edges of length a.



Use separation of variables. Write

$$\begin{aligned} \phi(x,y) &= X(x)Y(y)\ , \\ Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} &= 0\ , \\ \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda. \end{aligned}$$

We thus have the two ODEs to solve:

$$\begin{aligned} \frac{d^2X}{dx^2} - \lambda X &= 0\ , \\ \frac{d^2Y}{dy^2} + \lambda Y &= 0\ . \end{aligned}$$

Then

$$\lambda < 0 \quad \begin{cases} X & \text{is sinusoidal} \\ Y & \text{is exponential} \end{cases} , \qquad \lambda > 0 \quad \begin{cases} X & \text{is exponential} \\ Y & \text{is sinusoidal} \end{cases}$$

The choice is determined by the **boundary conditions**, and for this example we choose $\lambda > 0$ and set $\lambda = m^2$. The separable solution is then

$$X_m(x) = Ae^{mx} + Be^{-mx}\ , \qquad Y_m(y) = C\sin(my) + D\cos(my).$$

We need

$$\phi(x,0) = \phi(x,a) = 0 \;, \quad \Longrightarrow \quad X(x)Y(0) = X(x)Y(a) = 0 \;, \quad 0 \le x \le a \;.$$

Hence, we impose

$$\begin{aligned} Y(0) &= Y(a) = 0 \;, \qquad \Longrightarrow \\ Y(y) &\equiv Y_n(y) = \sin\left(\frac{n\pi}{a} y\right) \;, \qquad n = 1, 2, \ldots \end{aligned}$$

So we have that the **allowed** values of m are $m = n\pi/a$ for $n = 1, 2, \ldots$. This in turn determines the $X_m(x)$ to have these values of m. Thus, the most **general** solution I can write down subject to these restrictions is

$$\phi(x,y) = \sum_{n=1}^{\infty} \left(A_n\, e^{n\pi x/a} + B_n\, e^{-n\pi x/a}\right) \sin\left(\frac{n\pi}{a} y\right) \;.$$

I can do this because the PDE is **linear**. [Note that these solutions are $\mathrm{Imag}(e^{\pm n\pi z/a})$, $z = (x + iy)$.]

We impose the boundary condition that

- $\phi(0,y) = 0$

  This gives $A_n + B_n = 0 \;, \; \forall\; n$.

- $\phi(a,y) = T(y)$

  This gives the **Fourier series**

  $$\sum_{n=1}^{\infty} 2A_n \sinh(n\pi) \sin\left(\frac{n\pi}{a} y\right) = T(y) \;,$$

  and hence that

  $$2A_n \sinh(n\pi) = \frac{2}{a} \int_0^a dy\, T(y) \sin\left(\frac{n\pi}{a} y\right) \;.$$

  Suppose, $a = 1, \quad T(y) = y$. Then

  $$2A_n \sinh(n\pi) = 2 \int_0^1 dy\, y \sin(n\pi y) \;.$$

  Now,

  $$\int_0^1 dy\, y \sin(py) = -\frac{d}{dp} \int_0^1 dy \cos(py) = -\frac{d}{dp}\left(\frac{\sin(p)}{p}\right) = \frac{1}{p^2}\sin(p) - \frac{1}{p}\cos p \;.$$

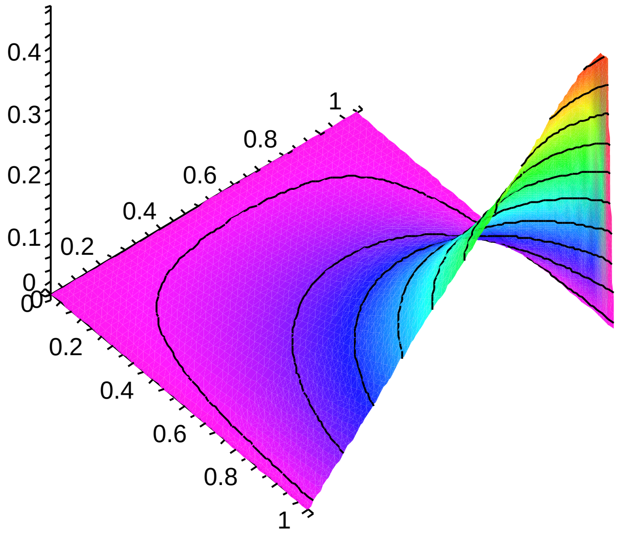
  Setting $p = n\pi$ we find

  $$A_n = -\frac{\cos(n\pi)}{n\pi \sinh(n\pi)} = \frac{(-1)^{n+1}}{n\pi \sinh(n\pi)} \;.$$

So

$$\phi(x, y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi \sinh(n\pi)} \sinh(n\pi x) \sin(n\pi y) \; .$$

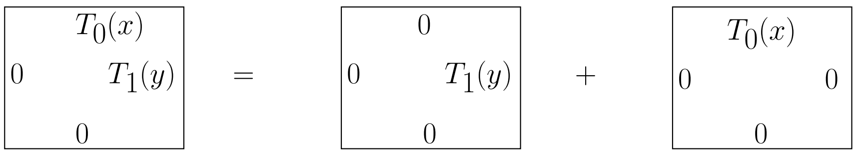
(This is a half-range sine series.)



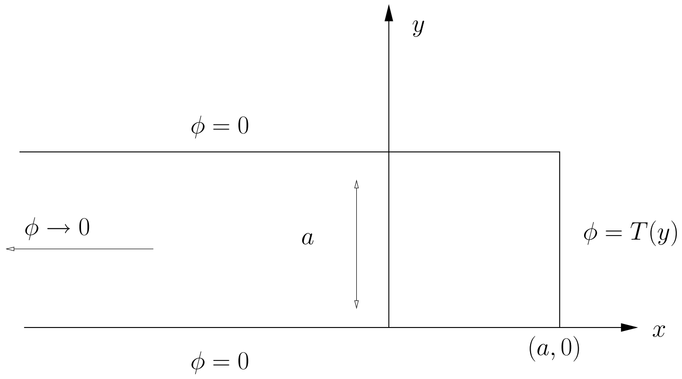
I used a sum of 100 terms here.

[Note that the set of separable solutions $\psi_n(x, y) = X_n(x)Y_n(y),\ n = 1, 2, \ldots$ form a **basis** for an ∞-dimensional vector space of which $\phi(x, y)$ is a member given by the **linear combination** of the basis vectors shown in the Fourier series.]

To solve for more complicated boundary conditions I can write the full solution by adding or **superposing** the solutions to related problems that we have already solved. E.g.,



Another example is the infinite strip of width a and with boundary conditions $\phi = 0$ on upper and lower edges and $\phi \to 0$ as $x \to -\infty$. On edge at $x = a$ have $\phi(a, y) = T(y)$.



The most general solution is then as before

$$\phi(x, y) \;=\; \sum_{n=1}^{\infty} \left(A_n\, e^{n\pi x/a} + B_n\, e^{-n\pi x/a}\right) \sin\left(\frac{n\pi}{a} y\right)$$

The condition $\phi \to 0$ as $x \to -\infty$ implies that $B_n = 0$.

On edge at $x = a$ we have

$$\phi(a, y) \;=\; \sum_{n=1}^{\infty} A_n\, e^{n\pi} \sin\left(\frac{n\pi}{a} y\right) \;=\; T(y)\;,$$

and so again have Fourier series which then gives

$$A_n \;=\; \frac{2}{a} e^{-n\pi} \int_0^a dy\, T(y) \sin\left(\frac{n\pi y}{a}\right)\;.$$

2.6 The wave equation

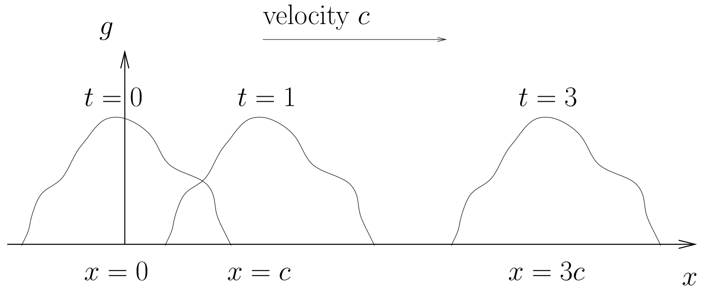
$$\frac{\partial^2 y}{\partial x^2} \;=\; \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}\;.$$

2.6.1 The infinite string

We found a solution in the form

$$y(x, t) \;=\; f(x + ct) + g(x - ct)\;.$$

Suppose that the function $g(v)$ is given. Then as t varies $g(x - ct)$ vs x looks like



The shape moves **right** with velocity c.

Likewise, the shape described by $f(u)$, plotted as a function of x, moves **left** with velocity c as t increases.

It is sufficient to know $f(x)$ and $g(x)$ for **all** x at $t = 0$ to specify the solution. The shapes are **unchanged**; they do not **disperse** but just move left and right, respectively, at velocity c: they are travelling waves. To find f and g it is sufficient to know the **initial conditions**

$$y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = v_0(x) \qquad \text{for all} \quad x.$$

Now, with $u = x + ct$ and $v = x - ct$,

$$\left.\frac{\partial f(u)}{\partial t}\right|_{t=0} \;=\; \left.\frac{\partial u}{\partial t}\frac{df}{du}\right|_{t=0} \;=\; c\left.\frac{df}{du}\right|_{t=0} \;=\; c\frac{df}{dx}\,.$$

Similarly,

$$\left.\frac{\partial g(v)}{\partial t}\right|_{t=0} \;=\; -\,c\frac{dg}{dx}\,.$$

The initial conditions are

(i) $f(x) + g(x) \;=\; y_0(x)$

(ii)

$$\begin{aligned} c\left(\frac{df}{dx} - \frac{dg}{dx}\right) \;&=\; v_0(x)\,, \quad \Longrightarrow \\ f(x) - g(x) \;&=\; \frac{1}{c}\int^x dx'\, v_0(x')\,. \end{aligned}$$

Thus

$$\begin{aligned} f(x) \;&=\; \frac{1}{2}y_0(x) + \frac{1}{2c}\int^x dx'\, v_0(x')\,, \\ g(x) \;&=\; \frac{1}{2}y_0(x) - \frac{1}{2c}\int^x dx'\, v_0(x')\,. \end{aligned}$$

So we have

$$\begin{aligned} y(x,t) \;&=\; f(x+ct) + g(x-ct) \\ &=\; \frac{1}{2}[y_0(x+ct) + y_0(x-ct)] \;+\; \frac{1}{2c}\int_{x-ct}^{x+ct} dx'\, v_0(x')\,. \end{aligned}$$

An example is

$$y_0(x) \;=\; e^{-x^2/2}\,, \qquad v_0(x) \;=\; x^2\,.$$

For which

$$y(x,t) \;=\; \frac{1}{2}[e^{-(x+ct)^2/2} + e^{-(x-ct)^2/2}] \;+\; \frac{1}{6c}[(x+ct)^3 - (x-ct)^3]\ .$$

The important special case mentioned earlier is of sinusoidal travelling waves:

$$y(x,t) \;=\; A\sin\big[\omega(x/c+t)\big] + B\sin\big[\omega(x/c-t)\big]\ .$$

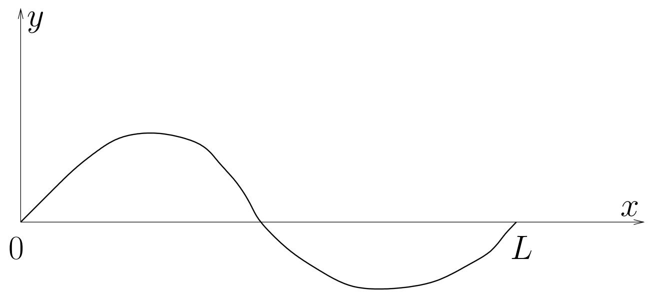
Here ω is the **frequency** of the wave.

It is often convenient to use complex exponential notation of the form

$$y(x,t) \;=\; Ae^{i\omega(x/c+t)} \;+\; Be^{i\omega(x/c-t)}\ ,$$

and use either the real or imaginary part of y at the end. Here A and B may be complex.

2.6.2 Finite string stopped at $x=0$ and $x=L$



The boundary conditions are

$$\begin{aligned} y(0,t) \;&=\; y(L,t) \;=\; 0, \qquad \text{for all } \ t\ , \\ y(x,0) \;&=\; y_0(x) \ \text{ and } \ \frac{\partial y}{\partial t}(x,0) \;=\; v_0(x) \qquad \text{for } \ 0<x<L. \end{aligned}$$

This will be solved by a Fourier series, and so we look for a separable solution:

$$y(x,t) \;=\; X(x)T(t)\ ,$$

giving

$$\frac{1}{X}\frac{d^2X}{dx^2} \;=\; \frac{1}{c^2T}\frac{d^2T}{dt^2} \;=\; \lambda\ .$$

We shall choose $\lambda = -m^2$ so that we obtain sinusoidal solutions. Then

$$\frac{d^2X}{dx^2} + m^2X \;=\; 0\ , \qquad \frac{d^2T}{dt^2} + c^2m^2T \;=\; 0\ ,$$

and we find the separable solution consistent with $y(0,t) \;=\; y(L,t) \;=\; 0$ to be

$$y_n(x,t) \;=\; \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \;, \qquad n = 1, 2, \ldots$$

The choice is $m = n\pi/L,\ n = 1, 2, \ldots$. Clearly, the solution vanishes at $x = 0, L\ \forall\ t$ by choice of the "sine" solution for $X(x)$.

The general solution is then, as before,

$$y(x,t) \;=\; \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \;.$$

The boundary conditions then impose two Fourier series

1.) $y(x,0) \;=\; y_0(x)$.

Set $t = 0$ to get

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \;=\; y_0(x) \;,$$

which gives

$$A_n \;=\; \frac{2}{L} \int_0^L dx\, y_0(x) \sin\left(\frac{n\pi x}{L}\right) \;.$$

2.) $\dfrac{\partial y}{\partial t}(x,0) \;=\; v_0(x)$.

First differential w.r.t. t and then set $t = 0$. We get

$$\sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \;=\; v_0(x) \;,$$

which gives

$$B_n \;=\; \frac{2}{n\pi c} \int_0^L dx\, v_0(x) \sin\left(\frac{n\pi x}{L}\right) \;.$$

The frequency of vibration for the separable solution labelled with n is $\omega_n = n\pi c/L$. The general solution consists of a superposition of modes, or harmonics, with allowed frequencies ω_n only.

2.7 The heat or diffusion equation

The diffusion equation in 1D is

$$\frac{\partial \Theta}{\partial t} \;=\; \kappa \frac{\partial^2 \Theta}{\partial x^2} \;,$$

with diffusivity $\kappa > 0$. Here $\Theta(x,t)$ is the concentration, or density, of material at time t. We expect the **total** amount of material to be conserved. I.e.,

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\, \Theta(x,t) \;=\; 0\;.$$

Taking the derivative under the integral sign and using the diffusion equation we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\, \Theta(x,t) \;&=\; \kappa\int_{-\infty}^{\infty} dx\, \frac{\partial^2\Theta(x,t)}{\partial x^2}\\ &=\; \left[\frac{\partial\Theta(\infty,t)}{\partial x} - \frac{\partial\Theta(-\infty,t)}{\partial x}\right]\;.\end{aligned}$$

The second step is integration by parts **at fixed** t or just simply integration of a derivative. For any **physical** distribution we assume

$$\frac{\partial\Theta(\pm\infty,t)}{\partial x} \;=\; 0\;,$$

i.e., there is no outward flux at $x = \pm\infty$. Hence, RHS = 0 and the material is conserved. [This kind of manipulation is important in quantum mechanics where total probability is conserved.]

2.7.1 infinite bar

Look for solutions in terms of the dimensionless variable

$$u \;=\; \frac{x}{(4\kappa t)^{1/2}}\;,$$

of the form

$$\Theta(x,t) \;=\; F(u)\;.$$

We have

$$\begin{aligned}\frac{\partial\Theta}{\partial x} \;&=\; F'(u)\frac{\partial u}{\partial x} \;=\; F'(u)(4\kappa t)^{-1/2}\;,\\ \frac{\partial^2\Theta}{\partial x^2} \;&=\; F''(u)(4\kappa t)^{-1}\;,\\ \frac{\partial\Theta}{\partial t} \;&=\; F'(u)\frac{\partial u}{\partial t} \;=\; F'(u)\left(\frac{-u}{2t}\right)\;.\end{aligned}$$

The heat/diffusion equation them becomes

$$\kappa F''(u)(4\kappa t)^{-1} \;=\; -\frac{u}{2t}F'(u)\;,$$

or

$$\frac{F''(u)}{F'(u)} \;=\; -\,2u\;.$$

This is

$$\begin{aligned}\frac{d}{du}\,\ln F'(u) &= -2u\;, \qquad \Longrightarrow \\ F'(u) &= Ae^{-u^2}\;.\end{aligned}$$

We thus find that

$$F(u) \;=\; A\int_0^u ds\, e^{-s^2} \;+\; B\;.$$

Choose $A = 1/\pi^{1/2}$ and $B = 1/2$. We define the **error** function $\mathrm{erf}\,(u)$ by

$$\mathrm{erf}\,(u) \;=\; \frac{2}{\pi^{1/2}}\int_0^u ds\, e^{-s^2}\;,$$

which satisfies

$$\mathrm{erf}\,(\pm\infty) \;=\; \pm\,1 \qquad\qquad \text{since} \qquad\qquad \int_0^\infty ds\, e^{-\alpha s^2} \;=\; \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\;.$$

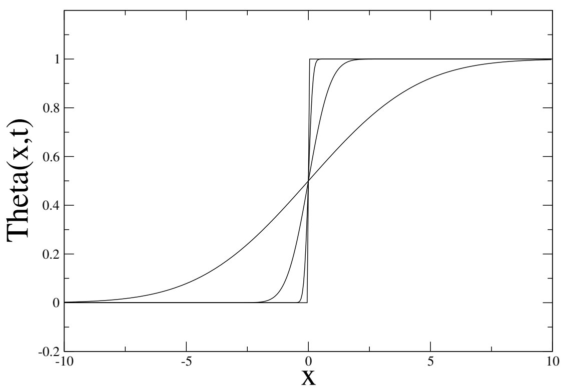
Then we have the solution

$$\Theta(x,t) \;=\; \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{4\kappa t}\right)\right)\;.$$

For t small and $x > 0$ ($x < 0$) the argument, u, approaches $u = \infty$ ($-\infty$). Since $\mathrm{erf}\,(\pm\infty) = \pm 1$, the **initial state** at $t = 0$ is

$$\Theta(x,0) \;=\; \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}\;.$$

The solution for $t \geq 0$ as t increases then looks like



2.7.2 ink drop

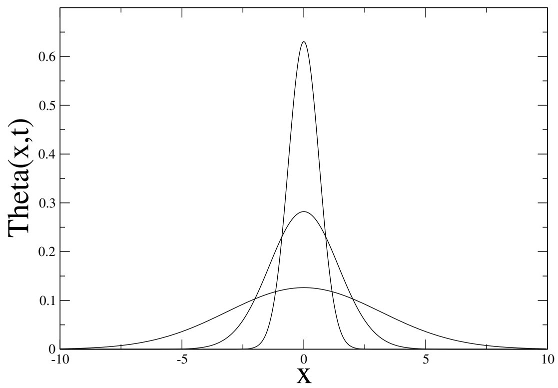
Note now that, if $\Theta_1(x,t)$ is a solution to the diffusion equation, then so is

$$\Theta(x,t) \;=\; \frac{\partial \Theta_1(x,t)}{\partial x} \,,$$

and hence taking $\Theta_1(x,t)$ to be the error function solution above we find the new solution

$$\Theta(x,t) \;=\; \frac{\partial}{\partial x}\frac{1}{\sqrt{\pi}}\int_0^{x/(4\kappa t)^{1/2}} ds\, e^{-s^2} \;=\; \frac{1}{(4\pi\kappa t)^{1/2}} e^{-x^2/4\kappa t} \,.$$

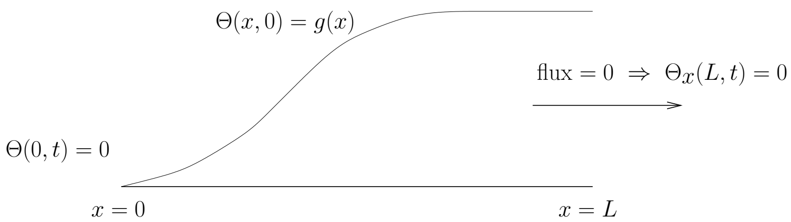
For t small the material is concentrated in a small region around $x = 0$ of width $x \sim (\kappa t)^{1/2}$. The region of high concentration increases in size as the material spreads out: it diffuses. The initial state is therefore a highly concentrated spot at $x = 0$; e.g., an ink drop dropped into water. In fact, this is a normal distribution and as t increases we get



2.7.3 bar of finite length

Consider a bar that occupies $0 \le x \le L$. The distribution of temperature obeys the heat equation.

In this example we give the boundary conditions as



- at $x = 0$ maintain temperature at zero: $\Theta(0,t) = 0 \quad t \ge 0$: Dirichlet condition;

- at $x = L$ apply insulating boundary condition: there is no flux of heat to $x > L$. Then have $\Theta_x(L,t) = 0, \quad t \geq 0$: Neumann condition;
- at $t = 0$ the initial distribution is given: $\Theta(x,0) = g(x), \quad 0 < x \leq L$: Dirichlet condition.

Look for separable solutions $\Theta(x,t) = X(x)T(t)$. Substituting find

$$\kappa X''(x)T(t) \;=\; X(x)T'(t) \;,$$

and so

$$\kappa \frac{X''(x)}{X(x)} \;=\; \frac{T'(t)}{T(t)} \;=\; C \quad \text{a constant.}$$

At this point in general we have different choices for C. These correspond to distinct physical situations and distinct kinds of boundary conditions. In our example we need $C < 0$. Set $C = -\kappa\alpha^2$. We have

$$\kappa \frac{X''(x)}{X(x)} \;=\; \frac{T'(t)}{T(t)} \;=\; -\,\kappa\alpha^2 \;,$$

with solutions

$$\begin{aligned} X(x) &= A\sin(\alpha x) + B\cos(\alpha x) \;, \\ T(t) &= D\,\exp(-\kappa\alpha^2 t) \;. \end{aligned}$$

- The condition $\Theta(0,t) = 0$ is satisfied by taking $B = 0$.
- The condition $\Theta_x(L,t) = 0$ then requires $X'(L) = 0$, and so

  $$\cos(\alpha L) \;=\; 0 \;, \qquad \Longrightarrow \qquad \alpha L \;=\; \left(n + \tfrac{1}{2}\right)\pi \;, \qquad n = 0, 1, \ldots, \infty.$$

  Note, that not all values of α are allowed.
- Can set $D = 1$ without loss of generality.

The most general solution satisfying the boundary conditions is then the Fourier series

$$\Theta(x,t) \;=\; \sum_{n=0}^{\infty} A_n \sin\left[\left(n + \tfrac{1}{2}\right)\pi x/L\right]\, \exp\left[-\kappa\left(n + \tfrac{1}{2}\right)^2 \pi^2 t/L^2\right] \;.$$

The initial condition now requires

$$\Theta(x,0) \;=\; \sum_{n=0}^{\infty} A_n \sin\left[\left(n + \tfrac{1}{2}\right)\pi x/L\right] \;=\; g(x) \;, \qquad 0 \leq x \leq L \;.$$

The functions $\left\{\sin\left[\left(n+\frac{1}{2}\right)\pi x/L\right], \quad n = 0, 1, \ldots\right\}$ are orthogonal. Therefore, multiplying both sides by $\sin\left[\left(m+\frac{1}{2}\right)\pi x/L\right]$ and integrating from 0 to L gives

$$A_m = \frac{2}{L}\int_0^L dx\, g(x) \sin\left[\left(m+\tfrac{1}{2}\right)\pi x/L\right] \ .$$

Remark on other choices for C

The choice $C > 0$ (real) in the separable solution discussed in the previous example will give solutions which grow exponentially in time and is thus unphysical. However, I **can** choose $C = i\omega$. This leads to **complex** separable solutions whose real and imaginary parts are then taken as the solutions for $\Theta(x,t)$. These are:

$$\begin{aligned}
\Theta(x,t) &= e^{-kx}\cos(\omega t - kx) \ , \\
\Theta(x,t) &= e^{-kx}\sin(\omega t - kx) \ , \\
\Theta(x,t) &= e^{kx}\cos(\omega t + kx) \ , \\
\Theta(x,t) &= e^{kx}\sin(\omega t + kx) \ ,
\end{aligned}$$

where k is an arbitrary constant and $\omega = 2k^2\kappa$. Can verify by substitution. Note that they represent **damped** travelling waves, the first two travelling in the **positive** x-dirn., and the second two in the **negative** x-dirn.

A physical application is to the temperature distribution interior to a bar subject to an oscillating heat source, of frequency ω, applied to one end. E.g., this is a simple model for the temperature in the interior of the earth subject to the daily cycle of radiation from the sun on its surface.

3 Elementary analysis

3.1 Introduction

Begin with a few definitions

3.1.1 functions

A function f is a rule which, for any given x, provides a value $f(x)$. In our context x and $f(x)$ will be real numbers.

(a) Sometime write $f : x \to y$ where y is real.

(b) f may not be defined for all x. E.g., $f(x) = \sqrt{x}$ is only defined for $x \geq 0$ if f is to be real.

(c) The **graph** of a function is a **set** of points in the plane $(x, y) : y = f(x)$. Not always easy to draw. E.g.,

$$f(x) \;=\; \begin{cases} 1/q & \text{if } x \text{ is rational with } \; x = p/q \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(d) A polynomial function of degree n takes the form

$$f(x) = a_0 x + a_1 x^2 + \ldots + a_n x^n \;,$$

where $a_0, \ldots, a_n$ are constants.

(e) A **rational function** has the form $f(x) = P(x)/Q(x)$ where P and Q are polynomials.

(f) **Algebraic functions** are evaluated by performing a **finite** number of mathematical operations $(+, -, \times, \ldots)$. E.g.,

$$F(x) \;=\; \frac{x^{1/4}\sqrt{8 - x^2} + 3x^5}{x^2 + 7} \;.$$

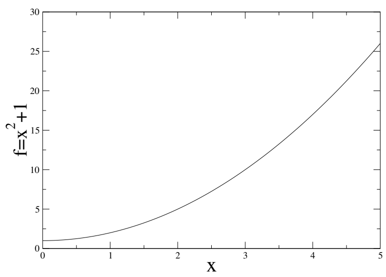
(g) **Transcendental functions** need an infinite number of operations. E.g.,

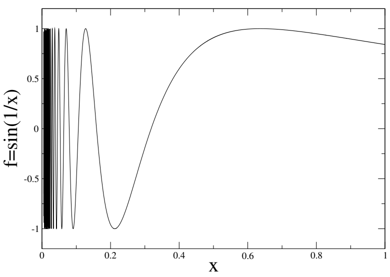
Exponential	$f(x)$	$=$	$a^x, \; a \neq 0, 1$
Logarithmic	$f(x)$	$=$	$\log_a x, a \neq 0, 1$
Trigonometric	$f(x)$	$=$	$\sin(x), \; \cos(x), \; \text{etc.}$
Hyperbolic	$f(x)$	$=$	$\sinh(x), \; \cosh(x), \; \text{etc.}$
Bessel function	$f(x)$	$=$	$J_\nu(x) \qquad \text{order} \;\; \nu$

3.2 Limits of functions and sequences

3.2.1 Definitions and examples

Consider the values of a function $f(x)$ as x gets closer and closer to x_0. E.g., as $x \to 0$ look at $f(x) = x^2 + 1$ and $f(x) = \sin(1/x)$:





In the first case $f(x) = x^2 + 1 \to 1$ as $x \to 0$. In the second case $f(x) = \sin(1/x)$ has no limit as $x \to 0$. The formal way to express this idea is as follows.

$$f(x) \to l \quad \text{as} \quad x \to x_0 \qquad \text{or} \qquad \lim_{x \to x_0} f(x) = l$$

if, given any $\epsilon > 0$ however small, we can find a δ depending on ϵ such that

$$|f(x) - l| < \epsilon \qquad \text{for all } x \text{ such that} \qquad |x - x_0| < \delta \; .$$

Note that it is not necessary that $f(x_0) = l$ since the point **at** $x = x_0$ is excluded from the definition. Examples are

(a)

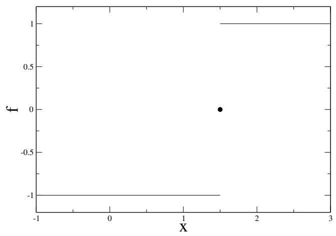
$$f(x) \;=\; \begin{cases} x^2 + 1 & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0. \end{cases}$$

Then $\lim_{x \to 0} f(x) = 1$ even though $f(0) = 0$; the limit exists but f is **discontinuous** at $x = 0$ since $f(0) \neq \lim_{x \to 0} f(x)$. Check the formal definition:

$$|f(x) - l| \equiv |x^2| < \epsilon \;\text{ if }\; |x - 0| \equiv |x| < \delta = \sqrt{\epsilon} \; .$$

(b)

$$f(x) \;=\; \begin{cases} -1 & x < a \\ 0 & x = a \\ 1 & x > a. \end{cases}$$



At any point x_0 other than $x_0 = a$ the limit of $f(x)$ as $x \to x_0$ exists:

$$\lim_{x \to x_0} f(x) \;=\; \begin{cases} -1 & x < a \\ 1 & x > a. \end{cases}$$

The limit as $x \to a$ does not exist. **However,** the "left-hand limit" and the "right-hand limit" **do** exist:

$$\text{as } \; x \to a \; \text{ from the left} \quad f(x) \to -1, \quad \text{written as} \quad \lim_{x \to a-} f(x) = -1 \; ,$$

$$\text{as } \; x \to a \; \text{ from the right} \quad f(x) \to 1, \quad \text{written as} \quad \lim_{x \to a+} f(x) = 1 \; .$$

(c) A **sequence** $\{u_n\}$ may be regarded as a function defined only at integer points $x = n$. E.g., $1, 1/3, 1/5, 1/7, \ldots : u_n = 1/(2n - 1)$. We normally only consider the limit of a sequence for $n \to \infty$.

The sequence has a limit l if, for any $\epsilon > 0$, there is an N depending on ϵ such that

$$|u_n - l| < \epsilon \qquad \text{for all} \qquad n > N \; .$$

3.2.2 properties of limits

If $\lim_{x\to x_0} a(x) = A$ and $\lim_{x\to x_0} b(x) = B$ then

(i) $\lim_{x\to x_0}[a(x) \pm b(x)] = A \pm B$.

(ii) $\lim_{x\to x_0}[a(x)b(x)] = AB$.

(iii) $\lim_{x\to x_0}[a(x)/b(x)] = A/B$ provided $B \neq 0$,

- if $A \neq 0$ and $B = 0$ then $\lim_{x\to x_0}[a(x)/b(x)]$ does not exist.
- if $A = 0$ and $B = 0$ then $\lim_{x\to x_0}[a(x)/b(x)]$ may or may not exist.
- if $A = \infty$ and $B = \infty$ then $\lim_{x\to x_0}[a(x)/b(x)]$ may or may not exist.

(iv) $\lim_{x\to x_0}[(a(x))^p] = A^p$ for any real number p, if A^p exists.

(v) $\lim_{x\to x_0}[(p)^{a(x)}] = p^A$ for any real number p, if p^A exists.

Examples are

$$\lim_{x\to\infty}\left[\frac{3x^2-5x}{5x^2+2x-6}\right] = \lim_{x\to\infty}\left[\frac{3x^2}{5x^2}\right] = \frac{3}{5}\,.$$

$$\begin{aligned} \lim_{x\to\infty}[\sqrt{x(x+1)} - x] &= \lim_{x\to\infty}[x(1+1/x)^{1/2} - x] = \\ \lim_{x\to\infty}[x(1+1/2x-1/8x^2\ldots) - x] &= \lim_{x\to\infty}[1/2 - 1/8x + \ldots] = 1/2. \end{aligned}$$

3.2.3 l'Hôpital's rule

If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$ then

$$\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to x_0}\left(\frac{f'(x)}{g'(x)}\right)\,,$$

whenever the second limit exists.

This result follows from Taylor's theorem. If f and g are differentiable n times on an interval containing x and x_0 then

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots + \frac{(x-x_0)^{n-1}}{(n-1)!}f^{(n-1)}(x_0) + R_n,$$

where $R_n = c\dfrac{(x-x_0)^n}{n!}f^n(\xi)$ for some ξ, $x_0 \le \xi \le x$. Similarly for $g(x)$. When $f(x_0) = g(x_0) = 0$ then

$$\begin{aligned}
\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) &= \lim_{x\to x_0}\left(\frac{(x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots}{(x-x_0)g'(x_0) + \frac{(x-x_0)^2}{2!}g''(x_0) + \ldots}\right) \\
&= \lim_{x\to x_0}\left(\frac{f'(x_0) + \frac{(x-x_0)}{2!}f''(x_0) + \ldots}{g'(x_0) + \frac{(x-x_0)}{2!}g''(x_0) + \ldots}\right) \\
&= \lim_{x\to x_0}\left(\frac{f'(x)}{g'(x)}\right) .
\end{aligned}$$

If $\lim_{x\to x_0} f'(x) = 0$ and $\lim_{x\to x_0} g'(x) = 0$ then the rule can be applied again, and so on until one of both of the limits is non-zero. In fact, just identify, respectively, the first term in each of the Taylor's series for f and g that does not vanish and inspect the ratio.

Examples are

(i)

$$\lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) = \lim_{x\to 0}\left(\frac{e^x}{1}\right) = 1 .$$

(ii)

$$\begin{aligned}
\lim_{x\to 0}\left[\frac{x\sin(x)}{\cos(x) - 1}\right] &= \lim_{x\to 0}\left[\frac{\sin(x) + x\cos(x)}{-\sin(x)}\right] \\
&= \lim_{x\to 0}\left[\frac{2\cos(x) - x\sin(x)}{-\cos(x)}\right] = -2 .
\end{aligned}$$

3.2.4 more examples

Some examples need **ad hoc** treatment. E.g.,

$$\lim_{x\to\infty}[x^2 e^{-x}] = \lim_{x\to\infty}\left[\frac{x^2}{e^x}\right] =$$

$$\lim_{x\to\infty}\left[\frac{x^2}{1 + x + x^2/2! + x^3/3! + \ldots}\right] \le \lim_{x\to\infty}\left[\frac{x^2}{1 + x + x^2/2! + x^3/3!}\right] = 0 ,$$

where we have assumed that the series expansion for e^x is valid (i.e., converges) for all x. We will come back to this later.

$$\lim_{x\to 0}\left[(\cos x)^{1/x^2}\right] = \lim_{x\to 0}\left[\exp\left(\frac{1}{x^2}\ln(\cos x)\right)\right] .$$

Use $\cos x = 1 - x^2/2! + x^4/4! + \ldots$ and $\ln(1+u) = u - u^3/3 + \ldots$. We find

$$\ln(\cos x) = -x^2/2! + O(x^4) \ ,$$

and so

$$\lim_{x \to 0} \left[(\cos x)^{1/x^2} \right] \ = \ \lim_{x \to 0} \left[\exp\left(-1/2 + O(x^2) \right) \right] \ = \ e^{-1/2} \ .$$

3.3 Series

3.3.1 Definitions

From a given sequence $u_1, u_2, u_3, \ldots$ we can form a new sequence $S_1, S_2, S_3, \ldots$, where

$$S_n \ = \ u_1 + u_2 + \ldots + u_n \ = \ \sum_{k=1}^{n} u_k \ .$$

The **infinite series**

$$\lim_{n \to \infty} S_n \ = \ \sum_{k=1}^{\infty} u_k,$$

if the limit exists, is of importance. S_n is called the nth partial sum. If $S = \lim_{n \to \infty} S_n$ exists, the series is said to be **convergent** with sum S. Otherwise, it is **divergent**. An example is the geometric series $S_n \ = \ a \sum_{k=1}^{n} r^k$.

- If $|r| < 1$ then $\lim_{n \to \infty} S_n = \dfrac{a}{1-r}$. The series is **convergent** with sum $S = \dfrac{a}{1-r}$.
- If $|r| \geq 1$, the series is **divergent**.

Some simple properties of series are

(a) Removal, insertion or alteration of a finite number of terms does not affect the divergence or convergence of a series.

(b) If $\sum_k u_k$ converges, then $\lim_{k \to \infty} u_k = 0$.

(c) If $\sum_k u_k$ converges, then so does the series obtained by bracketing the terms of $\{u_k\}$ in any manner to form a new series. The two series have the same sum. **Not** true for a divergent series. E.g.,

$$\sum_k (-1)^k \ = \ 1 + (-1+1) + (-1+1) + \ldots \ = \ 1$$

$$\sum_k (-1)^k \ = \ (1-1) + (1-1) + \ldots \ = \ 0 \ .$$

An important observation is

> If the sequence $S_1, S_2, S_3, \ldots$ is **increasing** and **bounded**, i.e., $S_n \geq S_m$ for all m, n with $n \geq m$, **and** there is a number B such that $S_n \leq B$ for all n, then the **sequence converges**.

Consider the geometric series above with $a = 1, r = 1/2$. Then

$$S_n = \frac{1 - 1/2^{n+1}}{1 - 1/2} = 2(1 - 1/2^{n+1}) \ .$$

Clearly, $\{S_n\}$ is an **increasing** sequence and $S_n \leq 2$ for all n: $B = 2$ ($B = 3$ would also do).

The limit of series with these properties is the **least upper bound** or **supremum** of the sequence. I.e., it is the smallest value of B that satisfies the criterion $S_n \leq B$. In the example, it is $B = 2$.

3.3.2 tests for convergence

First consider series with terms all of one sign (say positive).

(a) **comparison test**

> If $\sum_{k=1}^{\infty} v_k$ is convergent and $u_k \leq v_k$ for all $k \geq N$ for some N, then $\sum_{k=1}^{\infty} u_k$ is convergent.
>
> If $\sum_{k=1}^{\infty} v_k$ is divergent and $u_k \geq v_k$ for all $k \geq N$ for some N, then $\sum_{k=1}^{\infty} u_k$ is divergent.

Examples

(i) $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ is convergent since $\frac{1}{k2^k} \leq \frac{1}{2^k}$ for $k \geq 2$, and $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (a convergent geometric series).

(ii) $Z_1 = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges. To show this let $S_N = \sum_{k=1}^{N} \frac{1}{k}$. For any finite N we can group terms as follows:

$$S_N = (1) + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \ldots + \frac{1}{15}\right) + \ldots \ .$$

Each bracketed term is greater than 1/2. So, putting $N = 2^m - 1$, we have

$$S_{2^m-1} \;>\; \frac{1}{2} + \frac{1}{2} + \ldots \;=\; \frac{m}{2}, \quad k = 1, 2, \ldots .$$

Thus by taking N large enough, S_N can be made bigger than any chosen number. Hence, the series diverges.

(iii)

$$Z_p = \sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{diverges if } \; p \le 1 \\ \text{converges if } \; p > 1 \end{cases} .$$

$p = 1$: done already.

$p < 1$: diverges by comparison test since $k^{-p} \ge k^{-1}$ for $k > 1$.

$p > 1$ have

$$S_N \;=\; (1) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \ldots + \frac{1}{15^p}\right) + \ldots .$$

Now, for $m > 1$, the mth bracketed term is less than

$$\underbrace{\left(\frac{1}{(2^{m-1})^p} + \ldots + \frac{1}{(2^{m-1})^p}\right)}_{2^{m-1} \text{ terms}} \;=\; \frac{1}{2^{(m-1)(p-1)}} .$$

Thus in a similar manner to before

$$S_{2^m-1} \;<\; \frac{1}{1^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \ldots \frac{1}{2^{(m-1)(p-1)}} \;<\; \frac{1}{1 - 2^{1-p}} .$$

This is because the middle expression is a finite number of terms from the geometric series with $r = \dfrac{1}{2^{p-1}}$ and the last term on RHS is its infinite sum. Thus,

$$S_N \;<\; \frac{1}{1 - 2^{1-p}} \quad \text{for } \textbf{any} \;\; N ,$$

and the series converges (remember we have $p > 1$).

(a) **ratio comparison test**

Let $\displaystyle\lim_{k\to\infty} \frac{u_k}{v_k} = A$.

(i) If $A \ne 0$ or ∞, then $\sum_k u_k$ and $\sum_k v_k$ either both converge or both diverge.

(ii) If $A = 0$ and $\sum_k v_k$ converges then $\sum_k u_k$ converges.

(iii) If $A = \infty$ and $\sum_k v_k$ diverges then $\sum_k u_k$ diverges.

Example

$\sum_{k=1}^{\infty} u_k$ diverges for $u_k = \dfrac{4k^2 - k + 3}{k^3 + 2k}$. Let $v_k = \dfrac{1}{k}$. Then

$$\lim_{k\to\infty} \frac{u_k}{v_k} \;=\; 4\,,$$

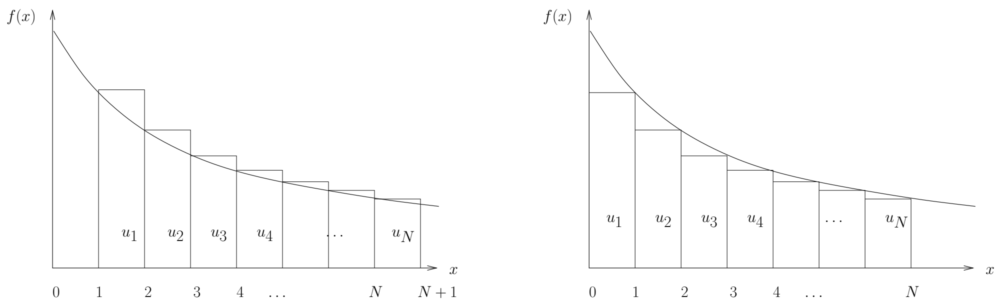
but $\sum_{k=1}^{\infty} v_k$ diverges and hence the result.

(c) **integral comparison test**

Suppose that $f(x) > 0$ is defined for all $x,\ 0 \le x \le \infty$, and is continuous and monotonically decreasing. Let $u_k = f(k)$. Then:

The series $\sum_{k=1}^{\infty} u_k$ converges if the integral $\int_0^{\infty} dx\, f(x)$ converges and diverges if integral $\int_1^{\infty} dx\, f(x)$ diverges. From the diagrams we see that

$$\int_1^{N+1} dx\, f(x) \le \sum_{k=1}^{N} u_k \le \int_0^{N} dx\, f(x)\ .$$



Both statements follow by letting $N \to \infty$.

(d) **D'Alembert's ratio test**

Let $\lim\limits_{k\to\infty} \dfrac{u_{k+1}}{u_k} = L$. Then

$$\sum_{k=1}^{\infty} u_k \ \text{ is } \ \begin{cases} \text{convergent if } \ L < 1\ , \\ \text{divergent if } \ L > 1\ . \end{cases}$$

The test is indecisive if $L = 1$.

For example, $\sum_{k=1}^{\infty} \dfrac{k}{2^k}$ is convergent since

$$\lim_{k\to\infty} \left[\frac{(k+1)/2^{k+1}}{k/2^k}\right] \;=\; \frac{1}{2}\ .$$

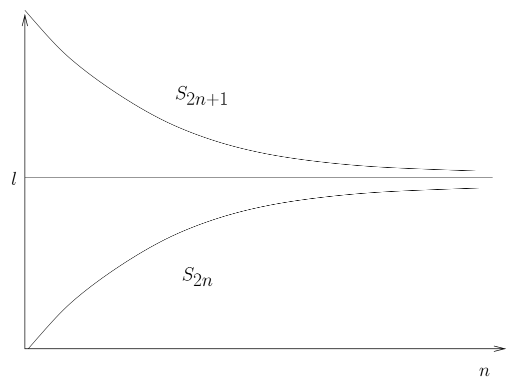
3.3.3 Alternating series

(a) If $\{u_k\}$ is a sequence of terms which have alternating + and − signs, and $|u_k|$ decreases monotonically to zero as $k \to \infty$ then $\sum_{k=1}^{\infty} u_k$ converges.

Assume w.l.o.g, that $u_1 > 0$. Then $S_1 > 0$ and $0 < S_2 < S_1$. We have

$$S_{2n+1} - S_{2n-1} = u_{2n} + u_{2n+1} < 0, \qquad S_{2n+2} - S_{2n} = u_{2n+1} + u_{2n+2} > 0 \, .$$

Thus, $\{S_{2n+1}\}$ form a **decreasing** sequence and $\{S_{2n}\}$ form an **increasing** sequence. Also, $S_{2n+1} - S_{2n} = u_{2n+1} \to 0$ as $n \to \infty$. Therefore, both tend to the same limit, l say.



Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \quad \text{converges to} \quad \ln(2) \, .$$

(b) Let $\{u_k\}$ be a sequence containing both positive and negative terms.

If $\sum_{k=1}^{\infty} |u_k|$ converges then $\sum_{k=1}^{\infty} u_k$ is said to be **absolutely convergent** and, hence, is also **convergent**.

If $\sum_{k=1}^{\infty} |u_k|$ diverges but $\sum_{k=1}^{\infty} u_k$ converges, the latter is said to be **conditionally convergent**. E.g., the example summing to $\ln 2$ in (a).

3.3.4 Power series

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots \, ,$$

is a power series about $x = 0$. Replace x by $(x - a)$ then it is a power series about $x = a$.

D'Alembert's ratio test shows that the series is absolutely convergent (and thus convergent) when

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| < 1\,, \quad \Longrightarrow \quad \lim_{k\to\infty}\left|\frac{c_{k+1}x}{c_k}\right| < 1\,.$$

This condition is satisfied when

$$|x| < R, \qquad \text{where} \qquad R = \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|\,.$$

It may or may not converge for $x = \pm R$. R is called the **radius of convergence**. For example,

$$x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}\,,$$

has radius of convergence $R = \lim_{k\to\infty}\dfrac{2k+3}{2k+1} = 1$.

3.3.5 Taylor series

From Taylor's theorem, if f is differentiable an infinite number of times on the inclusive interval from a to x, written $[a, x]$, then we have the series expansion

$$f(x) = \sum_{k=0}^{k} c_k(x-a)^k \qquad c_k = \frac{f^{(k)}(a)}{k!}\,.$$

The expansion is valid when the series converges. Examples:

(i) The Taylor's series for e^{-x} is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\,.$$

The radius of convergence is $R = \lim_{k\to\infty}\dfrac{(k+1)!}{k!} = \lim_{k\to\infty}(k+1) = \infty$. Thus, the series for e^{-x}, and similarly e^x, **always** converges and is **always** a good representation for this function. We used this earlier in an example.

(ii)

$$f(x) = \frac{1}{a+x}\,, \qquad f^{(k)}(0) \;=\; (-1)^k\frac{k!}{a^{k+1}} \quad \Longrightarrow \quad c_k \;=\; \frac{(-1)^k}{a^{k+1}}\,.$$

Thus, $R = \lim_{k\to\infty} a = a$. The series is absolutely convergent for $-a < x < a$.

Set $a = 1$.

At $x = -1$, $f(-1) = \infty$ and the series is $1 + 1 + 1 + \ldots$. It diverges.

At $x = 1$, $f(1) = 1/2$ and the series is $1 - 1 + 1 - \ldots$. It **still** diverges.