Statistical Field Theory

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Contents

1 BOOKS

(1) “Statistical Mechanics of Phase Transitions” J.M. Yeomans, Oxford Scientific Publications.

(2) “Statistical Physics”, Landau and Lifshitz, Pergammon Press.

(3) “The Theory of Critical Phenomena” J.J. Binney et al., Oxford Scientific Publications.

(4) “Scaling and Renormalization in Statistical Physics” John Cardy, Cambridge Lecture Notes in Physics.

(5) “ Quantum and Statistical Field Theory” M. Le Bellac, Clarendon Press.

(6) “Introduction to the Renormalisation Group of Critical Phenomena” P. Pfeuty and G. Toulouse, Wiley. ‘

(*7) “Fluctuation Theory of phase Transitions” Patashinskii and Pokrovskii, Pergamon.

(*8) “Field Theory the renormalisation Group and Critical Phenomena” D. Amit, McGaw-Hill.

(*9) “Quantum Field Theory of Critical Phenomena” J. Zinn-Justin, OXFORD.

(*10) “Phase Transitions and Critical Phenomena” in many volumes, edited by C. Domb and M. Green/Lebowitz.

(*11) “Statistical Field Theory” Vols I and II, Itzykson and Drouffe, CUP.

Note * means it’s a harder book.

2 Introduction

A general problem in physics is to deduce the macroscopic properties of a quantum system from a microscopic description. Such systems can only be described mathematically on a scale much smaller than the scales which are probed experimentally or on which the system naturally interacts with its environment. An obvious reason is that systems consist of particles whose individual behaviour is known and also whose interactions with neighbouring particles are known. On the other hand experimental probes interact only with systems containing large numbers of particles and the apparatus only responds to their large scale average behaviour. Statistical mechanics was developed expressly to deal with this problem but, of course, only provides a framework in which detailed methods of calculation and analysis can be evolved.

These notes are concerned with the physics of phase transitions: the phenomenon that in particular environments, quantified by particular values of external parameters such as temperature, magnetic field etc., many systems exhibit singularities in the thermodynamic variables which best describe the macroscopic state of the system. For example:

(i) the boiling of a liquid. There is a discontinuity in the entropy,

$$\Delta S = \frac{\Delta Q}{T_c}$$

where ΔQ is the latent heat. This is a first order transition;

(ii) the transition from paramagnetic to ferromagnetic behaviour of iron at the Curie temperature. Near the transition the system exhibits large-range cooperative behaviour on a scale much larger than the inter-atomic distance. This is an example of a second order, or continuous, transition. Scattering of radiation by systems at or near such a transition is anomalously large and is called **critical opalescence**. This is because the fluctuations in the atomic positions are correlated on a scale large compared with the spacing between neighbouring atoms, and so the radiation scattered by each atom is in phase and interferes constructively.

Most of the course will be concerned with the analysis of continuous transitions but from time to time the nature of first order transitions will be elucidated. Continuous transitions come under the heading of **critical phenomena**. It underpins the modern approach to non-perturbative studies of field theory and particularly **lattice-regularized field theories**.

Broadly, the discussion will centre on the following area or observations:

(i) the mathematical relationship between the sets of variables which describe the physics of the system on different scales. Each set of variables encodes the properties of the system most naturally on the associated scale. If we know how to relate different sets then we can deduce the large scale properties from the microscopic description. Such mathematical relationships are called, loosely, **renormalization group** equations, and , even more loosely, the relationship of the physics on one scale with that on another is described by the **renormalization group**. In fact there is no such thing as **the** renormalization group, but it is really a shorthand for the set of ideas which implement the ideas stated above and is best understood in the application of these ideas to particular systems. If the description of the system is in terms of a field theory then the renormalization group approach includes the idea of the **renormalization** of (quantum) field theories and the construction of **effective** field theories;

(ii) the concept of **universality**. This is the phenomenon that many systems whose microscopic descriptions differ widely nevertheless exhibit the **same** critical behaviour. That is, that near a continuous phase transition the descriptions of their macroscopic properties coincide in essential details. This phenomenon is related to the existence of fixed points of the renormalization group equations.

(iii) the phenomenon of **scaling**. The relationship between observables and parameters near a phase transition is best described by power-law behaviour. Dimensional analysis gives results of this kind but often the dimensions of the variables are **anomalous**. That is, they are different from the obvious or "engineering" dimensions. This phenomenon occurs particularly in low dimensions and certainly for $d < 4$. For example in a ferromagnet at the Curie temperature T_c we find

$$M \sim h^{\frac{1}{\delta}},$$

where M is the magnetization and h is the external magnetic field. Then the susceptibility, $\chi = \frac{\partial M}{\partial h}$, behaves like

$$\chi \sim h^{\frac{1}{\delta}-1}.$$

Since $\delta > 1$, χ **diverges** as $h \to 0$. The naive prediction for δ is 3. δ is an example of a **critical exponent** which must be calculable in a successful theory. The coefficients of proportionality in the above relations are **not** universal and are not easily calculated. However, in two dimensions the **conformal symmetry** of the theory at the transition point does allow many of these parameters to be calculated as well. We shall not pursue this topic in this course.

3 Definitions, Notation and Statistical Physics

All quantities of interest can be calculated from the partition function. We shall concentrate on classical systems although many of the ideas we shall investigate can be generalized to quantum systems. Many systems are formulated on a lattice, such as the Ising model, but as we shall see others which have similar behaviour are continuous systems such as H_20. However, it is useful to have one such model in mind to exemplify the concepts, and we shall use the Ising model in D dimensions to this end, leaving a more general formulation and notation for later.

The Ising model is defined on the sites of a D-dimensional cubic lattice, denoted Λ, whose sites are labelled by $\mathbf{n} = n_1\mathbf{e}_1 + \ldots n_D\mathbf{e}_D$, where the $\mathbf{e}_i,\ i = 1, \ldots, D$ are the basis vectors of a unit cell. With each site there is associated a spin variable $\sigma_\mathbf{n} \in (1, -1)$ labelled by $\mathbf{n}$. There is a nearest neighbour interaction and an interaction with an external magnetic field, h. Then the energy is written as

$$\begin{aligned} E(\{\sigma_\mathbf{n}\}) &= -J\sum_{\mathbf{n},\mu}\sigma_\mathbf{n}\sigma_{\mathbf{n}+\mu}, \quad J > 0 \\ H(\{\sigma_\mathbf{n}\}) &= E \ - \ h\sum_\mathbf{n}\sigma_\mathbf{n}, \end{aligned}$$

where μ is the lattice vector from a site to its nearest neighbour in the positive direction, i.e.,

$$\mu \in \mathbf{e}_1, \ldots, \mathbf{e}_D, \qquad \mathbf{e}_r = (0, \ldots, \underbrace{1}_{\substack{\text{r-th}\\ \text{posn}}}, \ldots, 0)\ . \tag{1}$$

Here $\{\sigma_\mathbf{n}\}$ is the notation for a **configuration** of Ising spins. This means a choice of assignment for the spin at each site of the lattice.

E is the energy of the nearest neighbour interaction between spins and the term involving h is the energy of interaction of each individual spin with the imposed external field h. H is the total energy. The coupling constant J can, in principle, be a function of the volume V since, in a real system, if we change V it changes the lattice spacing which generally will lead to a change in the interaction strength J. Note that h is under the experimenter's control and so acts as a probe to allow interrogation of the system.

We shall use a concise notation where the argument $\{\sigma_\mathbf{n}\}$ is replaced by σ. The partition function $\mathcal{Z}$ is defined by

$$\mathcal{Z} \ = \ \sum_\sigma \exp(-\beta H(\sigma)) \ = \ \sum_\sigma \exp(-\beta(E(\sigma) \ - \ h\sum_\mathbf{n}\sigma_\mathbf{n}))\ , \tag{2}$$

The system is in contact with a heatbath of temperature T with $\beta = 1/kT$. The way the expression for $\mathcal{Z}$ is written shows that h can be thought of as a chemical potential. We shall see that it is conjugate to the magnetization M of the system. The point is that the formalism of the grand canonical ensemble will apply. However, we can recover the results we need directly from (2).

We shall assume that J, the coupling constant in H, depends only on the volume V of the system. Then for a given T, V and h the equilibrium probability density for finding the system in configuration $\{\sigma_\mathbf{n}\}$ is

$$p(\sigma) \ = \ \frac{1}{\mathcal{Z}}\exp(-\beta H(\sigma)). \tag{3}$$

Averages over $p(\sigma)$ will be denoted with angle brackets:

$$\langle\, O(\sigma)\,\rangle \ = \ \sum_\sigma p(\sigma)\, O(\sigma)\ . \tag{4}$$

The entropy S is given by

$$\begin{aligned} S &= -k\sum_{\sigma} p(\sigma)\log(p(\sigma)) \\ &= -k\sum_{\sigma}\frac{1}{\mathcal{Z}}\exp(-\beta H)(-\beta H - \log\mathcal{Z}) \\ &= k(\beta(U - hM) + \log\mathcal{Z}), \end{aligned}$$

where

$$\begin{aligned} U &= \langle\, E\,\rangle && \equiv \text{ internal energy,} \\ M &= \langle \sum_{\mathbf{n}} \sigma_{\mathbf{n}}\,\rangle && \equiv \text{ magnetization.} \end{aligned} \tag{5}$$

Note that S, U, M are **extensive**. If two identical systems are joined to make a new system these variables double in value, that is, they scale with volume when all other thermodynamic variables are held fixed. **Intensive** variables such as density ρ, T, P, retain the same value as for the original system.

Then we have

$$kT\log\mathcal{Z} \;=\; -\,U + TS + hM \;=\; -\,F, \tag{6}$$

and hence

$$F \;=\; -\,\frac{1}{\beta}\log\mathcal{Z} \tag{7}$$

where F is the thermodynamic potential appropriate for T, V and h as independent variables. From now on we shall omit V explicitly since for the Ising model it plays no rôle of interest.

From (2) directly or from the usual thermodynamic properties of F we have

$$U - hM \;=\; -\left(\frac{\partial\log\mathcal{Z}}{\partial\beta}\right)_h, \qquad M \;=\; -\left(\frac{\partial F}{\partial h}\right)_T. \tag{8}$$

[Comment: These equations are the same as apply in the grand ensemble formalism for a gas with $h \leftrightarrow \mu$ and $M \leftrightarrow N$ where μ is the chemical potential and N is the number of molecules. The important point is that the external fields which we use to probe the system can be manipulated as general chemical potentials coupled to an appropriate thermodynamic observable which measures the response to changes in the probe field.]

We also have

$$\begin{aligned} \delta S &= -k\sum_{\sigma}\delta p\,\log p \;-\; k\sum_{\sigma} p\,\frac{1}{p}\,\delta p \\ &= -k\sum_{\sigma}\delta p\,(\,-\,\beta(E - h\sum_{\mathbf{n}}\sigma_{\mathbf{n}}) - \log\mathcal{Z} + 1)\,. \end{aligned}$$

Note:

$$\sum_{\sigma} p = 1 \;\Longrightarrow\; \sum_{\sigma}\delta p = 0\,. \tag{9}$$

Now

$$U = \sum_\sigma p\,E \;\Rightarrow\; \delta U = \sum_\sigma \delta p\,E \;+\; \underbrace{\sum_\sigma p\,\delta E}_{-P\,\delta V}\,, \tag{10}$$

and so we deduce the usual thermodynamic identity

$$dU = TdS - PdV + hdM\,. \tag{11}$$

We set $dV = 0$ from now on. In this case compare with the equivalent relation for a liquid-gas system (e.g., H_20):

$$\begin{aligned} dU &= TdS + hdM && \text{Ising model or ferromagnetic system} \\ dU &= TdS - PdV && \text{liquid-gas system.} \end{aligned}$$

In the latter case the density is $\rho = N/V$ where N is the (fixed) number of molecules. We may thus alternatively write

$$dU = TdS + \frac{NP}{\rho^2}d\rho\,, \tag{12}$$

and we see a strong similarity between the two systems with $M \leftrightarrow \rho$. We shall return to this shortly.

The system is translationally invariant and so

$$M = \langle \sum_{\mathbf{n}} \sigma_{\mathbf{n}} \rangle = N\,\langle \sigma_{\mathbf{0}} \rangle\,, \tag{13}$$

where N is the number of lattice sites and then $\langle \sigma_{\mathbf{0}} \rangle$ is the magnetization per site.

The susceptibility χ is defined by

$$\chi = \left(\frac{\partial M}{\partial h}\right)_T = -\left(\frac{\partial^2 F}{\partial h^2}\right)_T = kT\left(\frac{\partial^2 \log \mathcal{Z}}{\partial h^2}\right)_T\,. \tag{14}$$

From (2) we have

$$\begin{aligned} \chi/N &= \frac{\partial}{\partial h}\left(\frac{1}{\mathcal{Z}}\sum_\sigma \sigma_{\mathbf{0}}\, e^{-\beta\,(E - h\sum_{\mathbf{n}}\sigma_{\mathbf{n}})}\right) \\ &= \beta\left(\sum_\sigma p(\sigma)\,(\sigma_{\mathbf{0}}\sum_{\mathbf{n}}\sigma_{\mathbf{n}}) - \frac{kT}{\mathcal{Z}}\left(\frac{\partial \mathcal{Z}}{\partial h}\right)_T \sum_\sigma p(\sigma)\,\sigma_{\mathbf{0}}\right)\,. \end{aligned}$$

But

$$\frac{kT}{\mathcal{Z}}\left(\frac{\partial \mathcal{Z}}{\partial h}\right)_T = M = N\,\langle \sigma_{\mathbf{0}} \rangle\,, \tag{15}$$

and so we get

$$\begin{aligned} \chi/N &= \beta\left(\sum_{\mathbf{n}}\langle \sigma_{\mathbf{0}}\,\sigma_{\mathbf{n}} \rangle - N\langle \sigma_{\mathbf{0}} \rangle\langle \sigma_{\mathbf{0}}\rangle\right) \\ &= \beta\sum_{\mathbf{n}}\left(\langle \sigma_{\mathbf{0}}\,\sigma_{\mathbf{n}} \rangle - \langle \sigma_{\mathbf{0}} \rangle\langle \sigma_{\mathbf{0}} \rangle\right)\,. \end{aligned}$$

We define the correlation function

$$\begin{aligned} G(\mathbf{n},\mathbf{r}) &= \langle\, \sigma_{\mathbf{n}}\, \sigma_{\mathbf{n+r}} \,\rangle \;-\; \langle\, \sigma_{\mathbf{n}} \,\rangle \langle\, \sigma_{\mathbf{n+r}} \,\rangle \\ &= \langle\, \sigma_{\mathbf{n}}\, \sigma_{\mathbf{n+r}} \,\rangle_c \;, \end{aligned}$$

where the subscript c stands for connected. By translation invariance G is independent of $\mathbf{n}$ and we can write

$$\begin{aligned} G(\mathbf{r}) &= \langle\, \sigma_{\mathbf{0}}\, \sigma_{\mathbf{r}} \,\rangle \;-\; \langle\, \sigma_{\mathbf{0}} \,\rangle \langle\, \sigma_{\mathbf{0}} \,\rangle \;, \\ \Longrightarrow \quad \chi &= \beta N \sum_{\mathbf{r}} G(\mathbf{r}) \;. \end{aligned} \tag{16}$$

From its definition (16) is is reasonable to see that $G(\mathbf{r}) \to 0$ as $r = |\mathbf{r}| \to \infty$. This is because we would expect that two spins $\sigma_{\mathbf{0}}$ and $\sigma_{\mathbf{r}}$ will fluctuate independently when they are far apart and so their joint probability distribution becomes a product of distributions for the respective spins:

$$p(\sigma_{\mathbf{0}}, \sigma_{\mathbf{r}}) \longrightarrow p(\sigma_{\mathbf{0}}) p(\sigma_{\mathbf{r}}) \;\Longrightarrow\; \langle\, \sigma_{\mathbf{0}}\, \sigma_{\mathbf{r}} \,\rangle \longrightarrow \langle\, \sigma_{\mathbf{0}} \,\rangle \langle\, \sigma_{\mathbf{0}} \,\rangle \quad \text{as} \;\; r \to \infty \;. \tag{17}$$

Consider an external field $h_{\mathbf{n}}$ which depends also on $\mathbf{n}$. Then the magnetic interaction term is

$$-\sum_{\mathbf{n}} h_{\mathbf{n}}\, \sigma_{\mathbf{n}} \;. \tag{18}$$

If we follow the same algebra as for χ above we see that

$$\beta G(\mathbf{r}) \;=\; \left(\frac{\partial}{\partial h_{\mathbf{r}}} \right)_T \langle\, \sigma_{\mathbf{0}} \,\rangle \;. \tag{19}$$

Physically, $G(\mathbf{r})$ tells us the response of the average magnetization at site $\mathbf{0}$ to a small fluctuation in the external field at site $\mathbf{r}$. This is a fundamental object since it reveals in detail how the system is affected by external probes. We would expect $G(\mathbf{r}) \to 0$ as $r \to \infty$ since we expect the size of such influences to die away with distance. This should certainly be true for a local theory. We shall see that in many cases we can parameterize the large-r behaviour of G by

$$G(r) \;\sim\; \begin{cases} \dfrac{1}{r^{D-2+\eta}} & r \ll \xi \;, \\[2ex] \dfrac{\xi e^{-r/\xi}}{(\xi r)^{(D-1)/2}} & r \gg \xi \;, \end{cases} \tag{20}$$

where ξ is an important fundamental length in the theory called the **correlation** length. The exponent $(D-2+\eta)$ will be explained. The $(D-2)$ contribution can be deduced by dimensional analysis but η, which is an **anomalous** dimension is a non-trivial outcome of the theory.

For large $\xi \gg 1$ we have, from above,

$$\chi \;\propto\; \sum_{\mathbf{r}} G(r) \;\sim\; \int_{r<\xi} d^D r \, \frac{1}{r^{D-2+\eta}} \;=\; \xi^{2-\eta} \int_{u<1} d^D u \, \frac{1}{u^{D-2+\eta}} \;\sim\; \xi^{2-\eta} \;. \tag{21}$$

4 The D=1 Ising model and the transfer matrix

The Ising model is only soluble in D=1,2 and the $D = 2$ solution is a clever piece of analysis. To discuss the concepts which will be relevant to all the models we study it is useful to investigate the $D = 1$ model which can be used to highlight the ideas. Note that models in $D = 1$ are not trivial and many models have been studied in depth.

The expression for $\mathcal{Z}$ (2) in $D = 1$ can be written as follows

$$\mathcal{Z} \;=\; \sum_{\sigma_i=\pm1\;\forall i}\;\prod_{i=0}^{N-1} \exp\;\beta\left(J\sigma_i\sigma_{i+1} + \frac{1}{2}h\left(\sigma_i+\sigma_{i+1}\right)\right) \quad .$$

Note that the magnetic field term has been trivially rearranged. Now observe that this expression can be written as the trace over a product of N 2×2 matrices:

$$\mathcal{Z} \;=\; \sum_{\sigma_i=\pm1\;\forall i} W_{\sigma_0\sigma_1}W_{\sigma_1\sigma_2}W_{\sigma_2\sigma_3}\ldots W_{\sigma_{N-1}\sigma_0} \quad , \tag{22}$$

where the periodic boundary condition $\sigma_N = \sigma_0$ has been used. The matrix W is identified by comparing these two alternative ways of expressing $\mathcal{Z}$. We find

$$W_{\sigma\sigma'} \;=\; \exp\;\beta\left(J\sigma\sigma' + \frac{1}{2}h\left(\sigma+\sigma'\right)\right) \quad .$$

Evaluating with $\sigma, \sigma' = \pm1$ gives

$$W \;=\; \begin{pmatrix} \mu z & z^{-1} \\ z^{-1} & \mu^{-1}z \end{pmatrix} \quad , \tag{23}$$

where

$$\mu \;=\; e^B \qquad \text{and} \qquad z \;=\; e^K \quad B \;=\; \beta h \quad K \;=\; \beta J.$$

Thus from eqn. (22)

$$Z \;=\; \mathrm{Tr}\left(W^N\right) \quad ,$$

and hence

$$Z \;=\; \lambda_+^N \;+\; \lambda_-^N \quad ,$$

where λ_+ and λ_- are the eigenvalues of W with $\lambda_+ > \lambda_-$. For N large the first term dominates and we find that

$$Z \;=\; \lambda_+^N \quad .$$

Hence from eqn. (23) we have

$$\begin{aligned} &\lambda^2 \;-\; (2z\cosh B)\,\lambda \;+\; (z^2 - z^{-2}) \;=\; 0 \\ \Longrightarrow& \\ &\lambda_+ \;=\; \left[z\cosh B \;+\; \sqrt{z^2\sinh^2 B \;+\; z^{-2}}\right] \quad . \end{aligned} \tag{24}$$

W is know as the **transfer matrix** and is very important in many theoretical analyses. In higher dimensions it is a very large matrix indeed but a similar analysis goes through and the partition function is still given in terms of the **largest** eigenvalue. In fact, we need know only the few **largest** eigenvalues to determine all the observable thermodynamic variables. However, for a very large and even sparse matrix this can be a daunting task.

The free energy is then given by $F \;=\; -\,kTN\log\lambda_+$. Note that F is extensive i.e., it is proportional to N. This is, however, only true as $N \to \infty$ (otherwise λ_- contributes a term) which shows that the thermodynamic limit is necessary and N must be large enough that $(\lambda_-/\lambda_+)^N$ is negligibly small.

The magnetization per site M/N is given by

$$M/N \;=\; kT\left(\frac{\partial\,\log\lambda_+}{\partial h}\right)_T \;=\; \frac{e^K\,\sinh\,B}{\sqrt{e^{2K}\sinh^2\,B \;+\; e^{-2K}}}\;. \tag{25}$$

From now on we use M for magnetization per site, i.e., formerly M/N. In order to keep translation invariance but work with a finite but large number of spins N we shall use periodic boundary conditions. We may then consider the infinite volume limit $N \to \infty$.

The magnetization is given by

$$\begin{aligned} M \;&=\; \langle\,\sigma_p\,\rangle \;=\; \frac{1}{\mathcal{Z}}\sum_{\sigma} W_{\sigma_0\sigma_1}\,\ldots\,W_{\sigma_{p-1}\sigma_p}\,\sigma_p\,W_{\sigma_p\sigma_{p+1}}\,\ldots\,W_{\sigma_{N-1}\sigma_0}\;, \\ &=\; \frac{\mathrm{Tr}\,(W^p\,S\,W^{N-p})}{\mathrm{Tr}\,(W^N)} \;=\; \frac{\mathrm{Tr}\,(S\,W^N)}{\mathrm{Tr}\,(W^N)}\;. \end{aligned}$$

where S is the matrix

$$S \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\;,$$

We see explicitly that M is independent of the choice of p because of translation invariance encoded by the trace. Let

$$\begin{aligned} W\,\mathbf{e}_\pm \;&=\; \lambda_\pm\,\mathbf{e}_\pm\;, \\ P \;&=\; (\,\mathbf{e}_+,\;\mathbf{e}_-\,)\;. \end{aligned}$$

Then we write

$$W \;=\; P\,\Lambda\,P^{-1} \qquad \Lambda \;=\; \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}\;, \tag{26}$$

and so

$$M \;=\; \frac{\mathrm{Tr}\,(P^{-1}SP\Lambda^N)}{\mathrm{Tr}\,(\Lambda^N)}\;. \tag{27}$$

Now

$$\Lambda^N \;=\; \begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix} \;\Longrightarrow\; \mathrm{Tr}\,(\Lambda^N) \;=\; \lambda_+^N \;+\; \lambda_-^N\;.$$

P is an orthogonal matrix and has the form

$$P \;=\; \begin{pmatrix} \cos\,\phi & -\sin\,\phi \\ \sin\,\phi & \cos\,\phi \end{pmatrix}\;, \qquad \cot\,2\phi \;=\; e^{2K}\sinh\,B\;.$$

Then

$$P^{-1}SP \;=\; \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix}$$

and so from (27)

$$M \;=\; \frac{(\lambda_+^N \;-\; \lambda_-^N)\cos 2\phi}{\lambda_+^N \;+\; \lambda_-^N} \to \cos 2\phi \;\;\text{as}\;\; N \to \;\infty\;.$$

This is the result we have already derived in (25). In the limit $N \to \infty$

$$\Lambda^N \to \lambda_+^N \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\;,$$

a result we shall use below.

We now calculate $G(r)$ and so first look at the two spin expectation value:

$$\begin{aligned} \langle\, \sigma_0\, \sigma_r\, \rangle \;&=\; \frac{1}{\mathcal{Z}}\sum_{\sigma} \sigma_0 W_{\sigma_0\sigma_1}\;\ldots\; W_{\sigma_{r-1}\sigma_r}\,\sigma_r\, W_{\sigma_r\sigma_{r+1}}\;\ldots\; W_{\sigma_{N-1}\sigma_0}\;, \\ &=\; \frac{\mathrm{Tr}\,(SW^rSW^{N-r})}{\mathrm{Tr}\,(W^N)}\;, \end{aligned}$$

where we have used translation invariance. Then immediately

$$\langle\, \sigma_0\, \sigma_r\, \rangle \;=\; \frac{\mathrm{Tr}\,(P^{-1}SP\Lambda^rP^{-1}SP\Lambda^{N-r})}{\lambda_+^N} \;=$$

$$\mathrm{Tr}\left[\begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix}\begin{pmatrix} \lambda_+^r & 0 \\ 0 & \lambda_-^r \end{pmatrix}\begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix}\begin{pmatrix} \lambda_+^{-r} & 0 \\ 0 & 0 \end{pmatrix}\right], \tag{28}$$

where the $N \to \infty$ limit has been taken. Then have

$$\langle\, \sigma_0\, \sigma_r\, \rangle \;=\; \cos^2 2\phi \;+\; \sin^2 2\phi\, \left(\frac{\lambda_-}{\lambda_+}\right)^r \tag{29}$$

We define

$$\xi \;=\; 1/\log\,(\lambda_+/\lambda_-) \qquad \Longrightarrow$$

$$G(r) \;=\; \langle\, \sigma_0\, \sigma_r\, \rangle_c \;=\; \langle\, \sigma_0\, \sigma_r\, \rangle \;-\; \underbrace{\langle\, \sigma_0\, \rangle \langle\, \sigma_r\, \rangle}_{M^2} \qquad \Longrightarrow$$

$$G(r) \;=\; \sin^2 2\phi\, e^{-r/\xi} \tag{30}$$

which is an example of the behaviour quoted above (20) for $G(\mathbf{r})$. The important point is that in most models there is a unique correlation length and it is given by

$$\xi \;=\; 1/\log\,(\lambda_1/\lambda_2)$$

where $\lambda_1 \geq \lambda_2$ are the two largest eigenvalues of the transfer matrix. It is possible that there is more than one relevant correlation length e.g. $\xi' = 1/\log\,(\lambda_1/\lambda_3)$ but this depends on the physics being investigated and we shall not refer to this extension further. The **mass gap** m of the theory is the inverse correlation length $m = 1/\xi$. We shall see that a large class of the phase transitions we will be studying are connected with the limit $\xi \to \infty$ (or $m \to 0$). This, of course, means $\lambda_2 \nearrow \lambda_1$, i.e., the maximum eigenvalue of the transfer matrix is degenerate.

5 The Phenomenology of Phase Transitions

Statistical systems in equilibrium are described by macroscopic, thermodynamic, observables which are functions of relevant external parameters, e.g., temperature, T, pressure, P, magnetic field, h. These parameters are **external fields** (they may be $\mathbf{x}, t$ dependent) which influence the system and which are under the control of the experimenter.

the observables **conjugate** to these fields are:

entropy	S	conjugate to temperature	T
volume	V	conjugate to pressure	P
magnetization	M	conjugate to mag. field	h

Of course V and P may be swapped round: either can be viewed as an external field. More common thermodynamic observables are the specific heats at constant pressure and volume, respectively C_P and C_V; the bulk compressibility, K; and the energy density, ϵ.

Equilibrium for given fixed external fields is described by the minimum of the relevant **thermodynamic potential**:

				Legendre Transformation
Internal energy,	U	for fixed	S,V	
Helmholz free energy,	F	for fixed	T,V:	$F = U - TS, \quad T = (\partial U/\partial S)_V$
Gibbs free energy,	G	for fixed	T,P:	$G = F + PV, \quad P = -(\partial F/\partial V)_T$
Enthalpy	H	for fixed	S,P:	$H = U + PV, \quad P = -(\partial U/\partial V)_S$

A **phase transition** occurs at those values of the external fields for which one or more observables are singular. This singularity may be a discontinuity or a divergence. The transition is classified by the nature of the typical singularity that occur. Different **phases** of a system are separated by phase transitions.

Broadly speaking phase transitions fall into two classes:

(1) **1st order**

(a) Singularities are discontinuities.

(b) Latent heat may be non-zero.

(c) The correlation length is finite: $\xi < \infty$.

(d) Bodies in two or more different phases may be in equilibrium at the transition point. E.g.,

(i) the domain structure of a ferromagnet;

(ii) liquid-solid mixture in a binary alloy: the liquid is richer in one component than is the solid;

(e) the symmetries of the phases on either side of a transition are unrelated.

(2) **2nd and higher order: continuous transitions**

(a) Singularities are divergences. An observable itself may be continuous or smooth at the transition point but a sufficiently high derivative with respect to an external field is divergent. C.f., in a ferromagnet at $T = T_c$

$$M \sim h^{\frac{1}{\delta}}, \qquad \chi = \left(\frac{\partial M}{\partial h}\right)_T \sim h^{\frac{1}{\delta}-1}.$$

(b) There are no discontinuities in quantities which remain finite through the transition and hence the latent heat is zero.

(c) The correlation length diverges: $\xi \to \infty$.

(d) There can be no mixture of phases at the transition point.

(e) The symmetry of one phase, usually the low-T one, is a subgroup of the symmetry of the other.

An **order parameter**, Ψ, distinguishes different phases in each of which it takes distinctly different values. Loosely a useful parameter is a collective or long-range coordinate on which the singular variables at the phase transition depend. Ψ is generally an **intensive** variable.

In a ferromagnet the spontaneous magnetization **per unit volume** at zero field, $\mathbf{M}(T)$, is such an order parameter, i.e.,

$$\mathbf{M}(T) = \lim_{h \to 0+} \mathbf{M}(\mathbf{h}, T)$$

then $|\mathbf{M}(T)| = 0$ for $T \geq T_c$, and $|\mathbf{M}(T)| > 0$ for $T < T_c$.

Note: $\Im(T - T_c)^{\frac{1}{2}}$ will not do.

Ψ is not necessarily a scalar, but in general it is a tensor and is a field of the **effective field theory** which describes the interactions of the system on macroscopic scales (i.e., scales much greater than the lattice spacing). The idea of such effective field theories is common to many areas of physics and is a natural product of renormalization group strategies.

Examples

(1) **The ferromagnet in 3 dimensions.**

The ferromagnet can be modelled by the Ising model defined in the previous section.

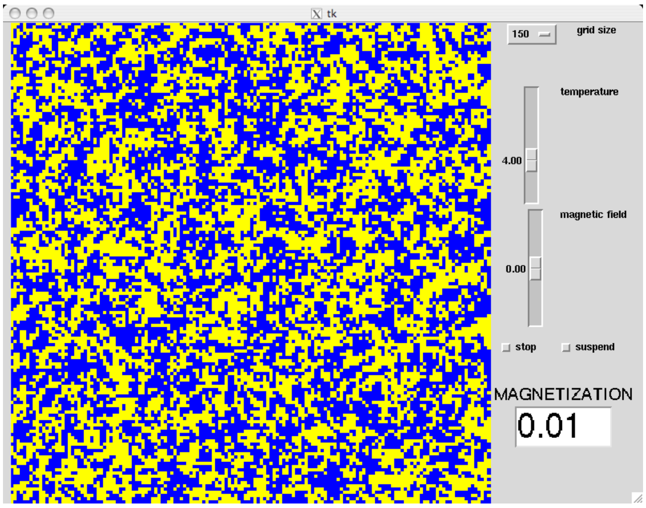
$$H \;=\; -J\sum_{\mathbf{n},\mu}\sigma_{\mathbf{n}}\sigma_{\mathbf{n}+\mu} \;-\; h\sum_{\mathbf{n}}\sigma_{\mathbf{n}},$$

The order parameter is the magnetization per unit volume.

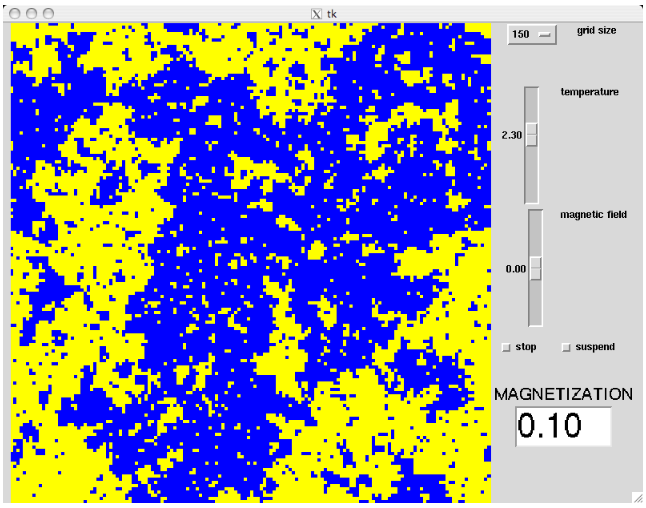
$$M = \frac{1}{V}\sum_{\mathbf{n}} \sigma_{\mathbf{n}}, \quad V = Na^3$$

where N is the number of sites in the lattice and a is the lattice spacing. Note, that whilst the $\sigma_{\mathbf{n}}$ are discrete, M is a continuous variable in the limit $N, V \to \infty$.

Note: from now on we use the symbol M now to mean magnetization **per unit volume**



Ising model in two dimensions. Well above the critical temperature $T_c = 2/\log(1+\sqrt{2}) \sim 2.27$. Note that the total magnetization is essentially zero. The theory is paramagnetic.



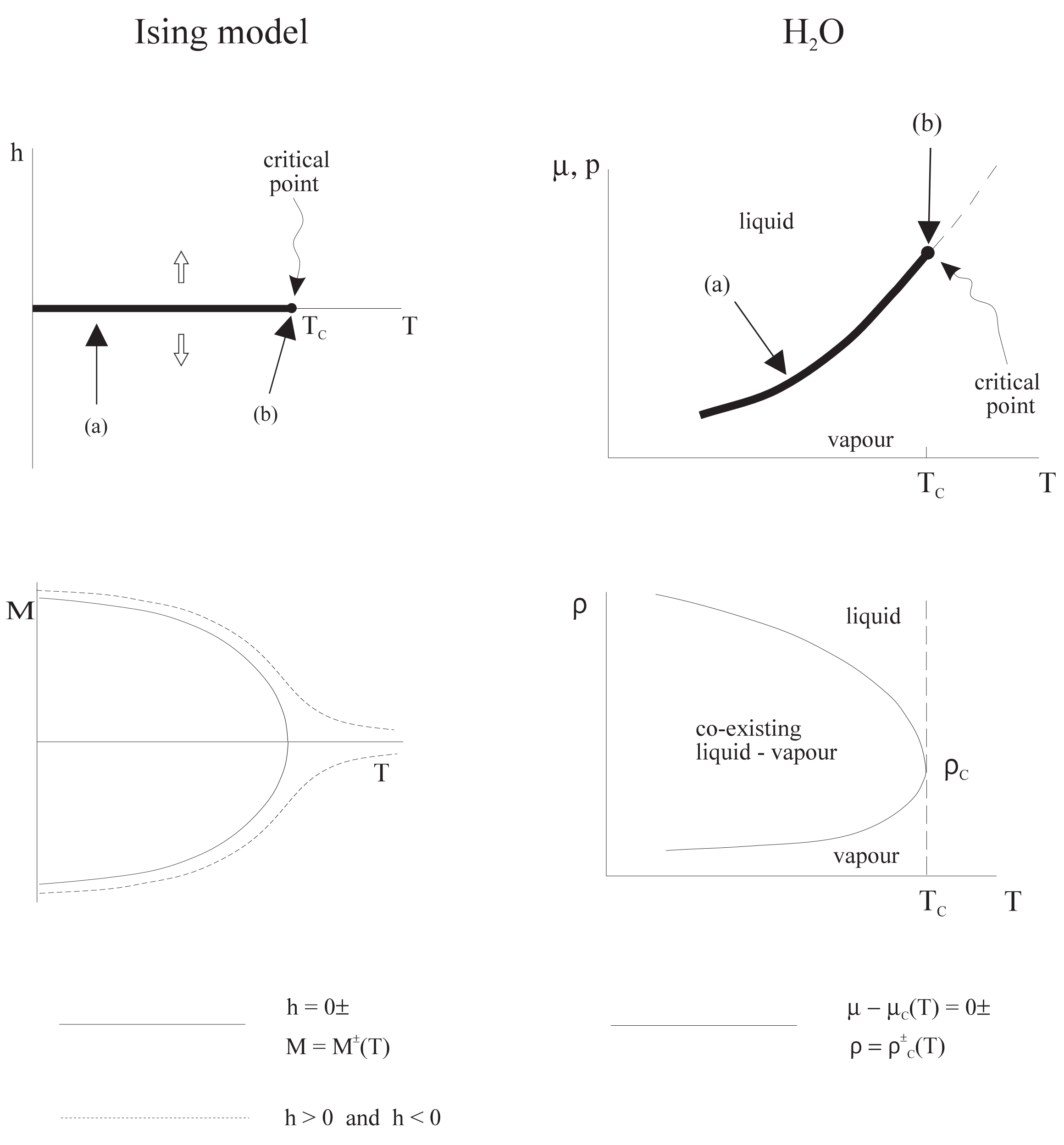
Ising model in two dimensions. Only just above the critical temperature $T_c \sim 2.27$. Note that the total magnetization is still close to zero but the sizes of the domains are larger. At the critical point there are domains of all sizes distributed according to a power law distribution.

(2) **H_2O**

Look at the two phases of liquid and vapour. The order parameter is the density, ρ, which is large for the liquid phase relative to its value for the vapour phase.

The properties of both systems, their similarities and differences are best exhibited by showing the various phase diagrams.

PHASE DIAGRAMS



μ is the chemical potential, and $\mu = \mu_c(T)$ is the line of first order transitions in the (μ, T) plot for H_20. It corresponds to the line $h = 0$ in the (h, T) plot for the Ising model. The critical point terminates the line of 1st order transitions.

(i) Approach along (a) gives a 1st order transition whilst approach along (b) through the **critical point** gives a 2nd order transition.

(ii) The order parameters are:

magnetization M	*density* ρ

The conjugate fields are:

magnetic field h	*chemical potential,* μ *or pressure,* P

(iii) The behaviour near $T = T_c$ $(t = (T - T_c)/T_c)$

(a) $\underline{t \to 0-,\ h = 0\pm}$ | $\underline{t \to 0-,\ \mu - \mu_c(T) = 0\pm}$

$M(T) \sim |t|^\beta$

$$\begin{aligned} \rho^*(T) &= \rho_c^+(T) - \rho_c \sim |t|^{\beta_+} \\ \rho^*(T) &= \rho_c^-(T) - \rho_c \sim |t|^{\beta_-} \end{aligned}$$

Clear symmetry in curve | No obvious symmetry but experimentally $\beta_+ = \beta_-$

(b) $\underline{t \to 0+,\ h = 0}$ | $\underline{t \to 0+,\ \mu = \mu_c(T)}$

Susceptibility

$\chi = \left(\frac{\partial M}{\partial h}\right)_T$

$\chi = \frac{K(T)}{K_0(T)}$
$K(T) = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T$
$K_0(T)$ is for ideal gas

Then

$$\chi(t) \sim |t|^{-\gamma}$$

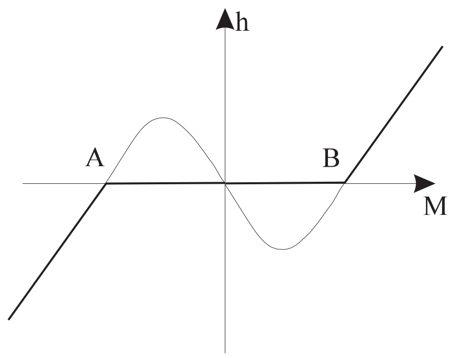
Note that for the Ising model with $t \to 0-,\ h \to 0+$ we find $\chi(T) \sim |t|^{-\gamma\prime}$ with $\gamma\prime = \gamma$. It should be remarked, howver, that $\gamma\prime$ is not defined for all models.

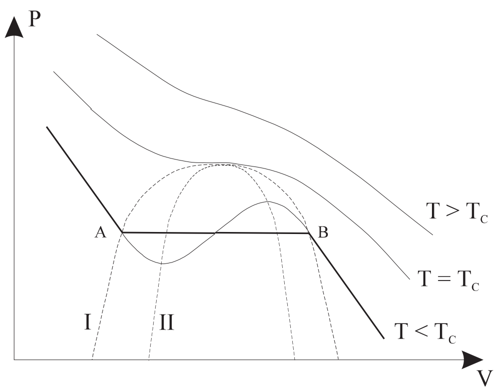
(c) $\underline{t = 0,\ h \to 0+}$ $\qquad$ $\underline{t = 0,\ \mu - \mu_c \to 0+}$

$M \sim h^{\frac{1}{\delta}}$ $\qquad$ $\rho - \rho_c \sim (\mu - \mu_c)^{\frac{1}{\delta}}$

β, γ, δ are examples of critical exponents.

(iv) Coexisting phases





(a) States between the curves I and II are physical but metastable. They do not violate thermodynamic inequalities. In the PV plot this is equivalent to

$$\left(\frac{\partial P}{\partial V}\right)_T < 0$$

which means that the compressibility is positive. This inequality is derived from entropy being a maximum in equilibrium. However, these states are unstable against changing to the mixed system, e.g., domains in the Ising model (or ferromagnet), and liquid-gas mixture for water.

The continuous curves shown are the (h, M) and (P, V) curves for a **pure phase**. E.g., the Van-der-Waals equation of state:

$$\left(P + \frac{a}{V}\right)(V - b) = cT$$

(b) The Maxwell construction gives the true equilibrium curve taking into account the formation of the mixed system. The mixture is of the two phases A and B. The rule for finding the interpolation is illustrated in the case of H_2O:

$$P_A = P_B, \quad \mu_A = \mu_B \Rightarrow \mu_A - \mu_B = \int_A^B v\ dP = 0,$$

where $v = V/N$, and N is the number of particles. This is the **equal areas rule** of Maxwell.

5.1 The general structure of phase diagrams

A thermodynamic space, Y, is some region in an s-dimensional real vector space spanned by field variables $y_1, \ldots, y_s$ (e.g., $P, V, T, \mu, \ldots$). In Y there will be points of two, three, etc. phase coexistence (c.f. A and B in H_2O plot above), together with critical points, multicritical points, critical end points, etc.. Q is the totality of such points. The **phase diagram** is the pair (Y, Q).

Points of a given type lie in a smooth manifold, M, say. The **codimension**, κ, of these points is defined by

$$\kappa = \dim(Y) - \dim(M).$$

E.g., two-phase points have $\kappa = 1$; critical points (points that terminate two phase lines) have $\kappa = 2$.

There do not exist any simple rules for constructing geometrically all acceptable phase diagrams, (Y, Q): we cannot easily construct all the phase diagrams which could occur naturally.

5.1.1 Structures in a phase diagram: a description of Q

I assume that there are c components in the system, and hence there are $(c+1)$ external fields: $\mu_1, \ldots, \mu_c, T$. Then $\dim(Y) = (c+1)$.

Manifolds of multiphase coexistence.

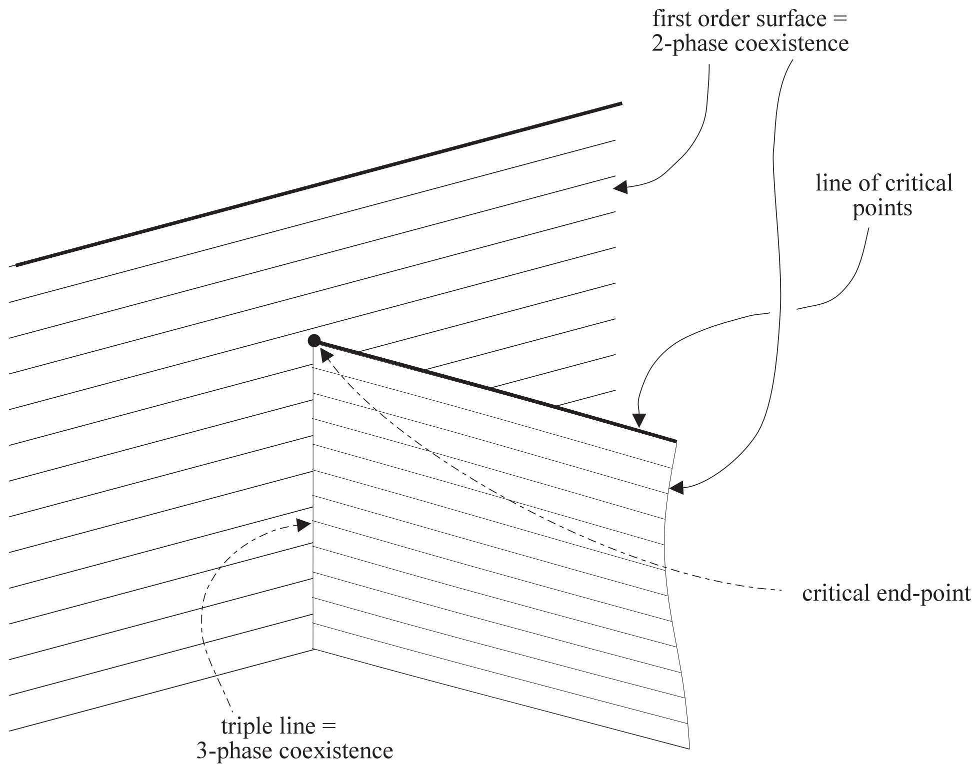
The **Gibbs phase rule** states that the coexistence of m phases in a system with C components has

$$f = c + 2 - m$$

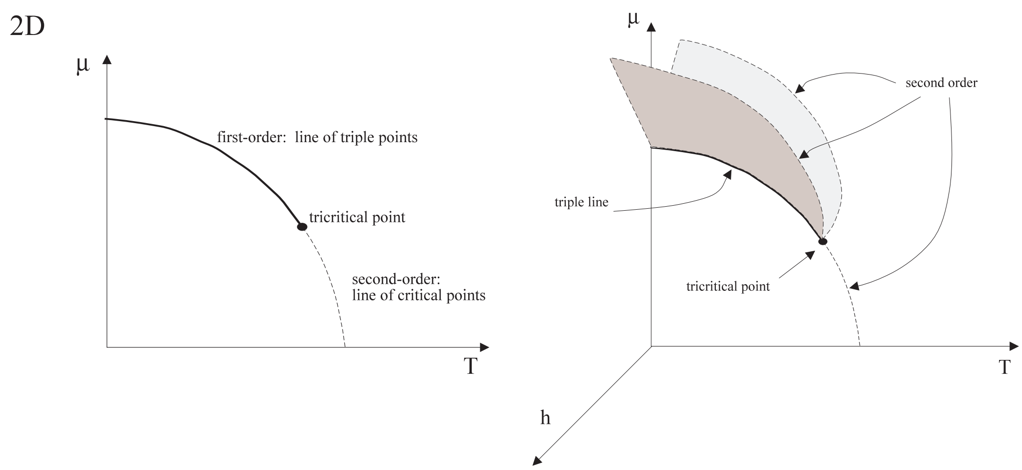
where f is the dimension of the manifold of m-phase coexistence.

proof: $\dim(Y) = (c+1)$ and hence the manifold has codimension $\kappa = (c+1-f)$. But $\kappa = (m-1)$ since κ external fields must be tuned to bring about m-phase coexistence.

An example of structures in a three dimensional phase diagram is shown below.



A **tricritical point** has $\kappa = 4$. Its nature is most easily seen first in three dimensions. This is already a special case since we can only be sure it will appear in four dimensions. We suppose we have taken the appropriate cross-section of the 4D space. this often occurs naturally since some of the parameters are naturally set to the special values necessary to show up the tricritical point: e.g., by symmetry considerations.



The hatched surfaces are 1st-order surfaces: surfaces of two phase coexistence. Thus the 1st order line in 2D is really a line of triple points (three phase coexistence) in higher dimensions.

6 Landau-Ginsburg theory and mean field theory

The Landau-Ginsburg theory is a phenomenological theory describing all types of phase transition which can be derived from the more complete theory. It is a classical approach which breaks down in its simple form for low dimensions. However, it can be used for developing the structure of phase transitions and phase diagrams. Landau theory gives the correct prediction for critical indices in dimensions $D > D_c$, where D_c is a critical dimension which is different for different kinds of critical point. E.g., for an ordinary critical point $D_c = 4$, and for a tricritical point $D_c = 3$.

Mean field theory is a method of analysing systems in which the site variable (spin etc.) is assumed to interact with the **mean field** of the neighbours with which it couples. In a spin model each of the neighbouring spins has the value of the mean magnetisation per spin, $\mathbf{M}$. The problem now reduces to that of a single spin in an external field and can easily be solved. By demanding that the mean value of the spin in question is $\mathbf{M}$ the solution yields a non-linear equation expressing this assumption of self-consistency and from which $\mathbf{M}$ can be calculated as a function of T. The approximation of the method is that it ignores fluctuations in the spins about their mean. It will turn out that Landau theory suffers from the same deficiency as we shall demonstrate. Mean field theory and Landau theory give the same, classical, predictions for critical exponents.

Let the order parameter be M. Expand the equilibrium free energy **per unit volume**, $A(T, M)$, as

$$A = A_0 + \frac{1}{2}A_2M^2 + \frac{1}{4}A_4M^4 + \frac{1}{6}A_6M^6 + \ldots, \tag{31}$$

with $A_2 \propto (T - T_c)$, $A_2 = a_2(T - T_c)$, when $|T - T_c|$ is small. Note that $A(T, M)$ is the appropriate thermodynamic potential when T, M are the independent variables.

There are no terms with odd powers of M. These can be present in principle but can be consistently excluded by symmetry considerations if the microscopic Hamiltonian is invariant under $M \to -M$. If odd powers of M are present then generally the theory has only first order transitions, although higher order transitions cannot be totally excluded. T_c is a complicated function of the couplings in the original, microscopic, Hamiltonian as are the other coefficients, A_{2n}. It is an assumption that A_2 is analytic in T: an assumption that can only be plausibly justified under certain circumstances. This assumption as well as others is wrong if the dimension is low enough.

Equilibrium is given by minimising the appropriate thermodynamic potential, in this case A:

$$\frac{dA}{dM} = 0, \qquad \textbf{The Equation of State.}$$

The observable value of the order parameter, $M(T)$, is the solution of this equation. Then

$$|M(T)| = \left|\frac{A_2}{A_4}\right|^{\frac{1}{2}} \left(1 + \frac{1}{2}\frac{A_6A_2}{A_4^2} + \ldots\right).$$

Thus as $T \to T_c$

$$|M(T)| \sim |T - T_c|^{\frac{1}{2}} \Rightarrow \beta = \frac{1}{2} \,.$$

We can rewrite the expression for $M(T)$ as

$$M(T) = \left|\frac{A_2}{A_4}\right|^{\frac{1}{2}} m(x), \qquad \text{where} \qquad x = \frac{A_6 A_2}{A_4^2}.$$

The analysis above is only possible if $A_4 > 0$, in which case A_6 only occurs in the correction terms. If $A_4 < 0$ then we require $A_6 > 0$ to stabilize the calculation and the results are different (see below).

$\mathbf{A_4 > 0}$

If a field h is applied then the symmetry is broken and

$$A = A_0 - hM + \frac{1}{2}A_2 M^2 + \frac{1}{4}A_4 M^4.$$

At $T = T_c$ ($A_2 = 0$) the condition for equilibrium is

$$-h + A_4 M^3 = 0 \quad \Rightarrow \quad M \sim h^{\frac{1}{3}} \quad \Rightarrow \quad \delta = 3.$$

For $T > T_c$ we have ($t = \frac{(T-T_c)}{T_c}$) the **Equation of State** (EoS) is

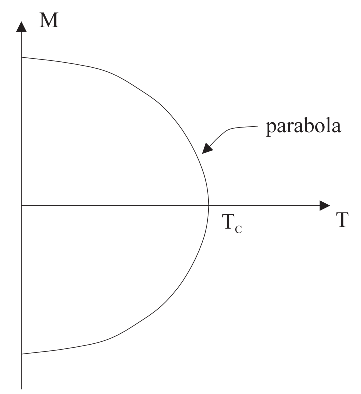
$$-h + a_2 T_c t M + A_4 M^3 = 0.$$

Then the susceptibility is given by

$$\chi = \left(\frac{\partial M}{\partial h}\right)_{h=0} = \frac{1}{a_2 T_c} t^{-1} \quad \Rightarrow \quad \gamma = 1.$$

Consider now $T < T_c$:

The curve of $\pm M(T)$ vs T for $h = 0$ is a parabola for sufficiently small t follows from the EoS



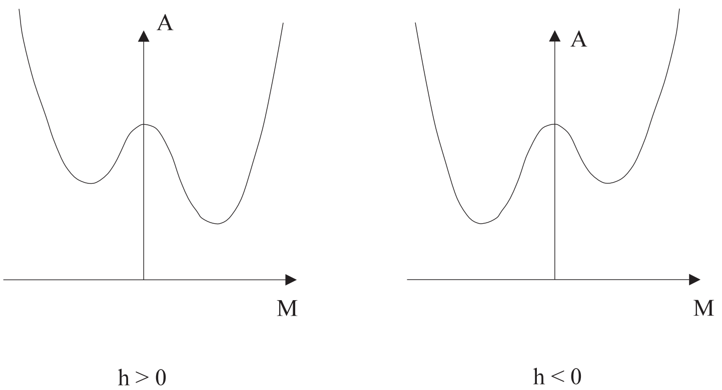
$$A_4 M^3 - a_2|T - T_c|M \sim h \ , \tag{32}$$

where A_4 is a function of T which does not vanish at $T = T_c$

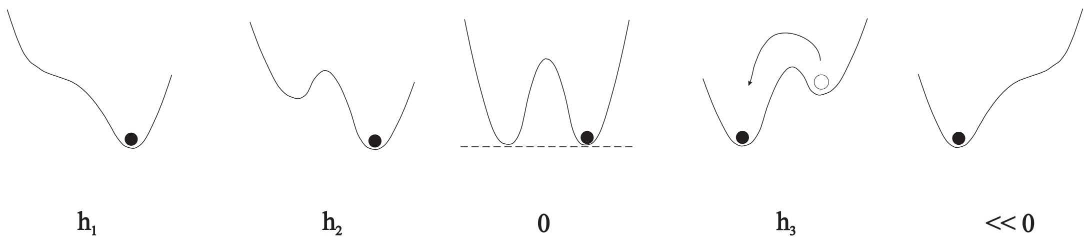
In general the EoS has the form

$$-a(T)M + b(T)M^3 + \ldots = h \tag{33}$$

with $a(T) > 0$. Then have



Plot A vs M for different h values to show what happens as h decreases from a large positive value and eventually becomes large and negative.

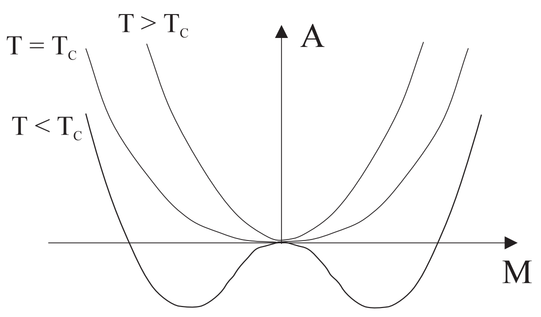


M varies smoothly as a function of h and metastability (supercooling) occurs when h becomes negative. Eventually, either a transition to the lowest state takes place or the metastable state becomes unstable and any fluctuation precipitates the change to the true equilibrium state.

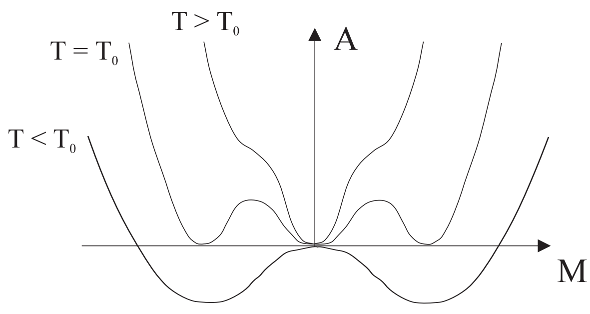
The **Maxwell construction** tells us that the stable state is not most generally characterised by a constant M. **Locally** M is a constant but it can change **globally** giving a domain structure. The Maxwell construction corresponds to the situation when $h = 0$ and there are two degenerate minima associated with the two different domains that can co-exist. Thus at $h = 0$ any value of M between these minima is possible and corresponds to an appropriate mixture of domains. The above analysis relies on the smoothness and differentiability of all functions A, h, M and can never directly address the mixed-phase system.

$\mathbf{A_4 \leq 0}$

As T decreases $A(M)$ behaves qualitatively differently depending on whether $A_4 > 0$ or $A_4 \leq 0$:



second-order transition at T = T_C



first-order phase transition at T_0 , T_0 > T_C

Hence the system passes from a second-order transition to a first-order transition as A_4 changes sign and becomes negative.

The stationary points are at $M = 0$ and at

$$M^2 = \left[-A_4 \pm \left(A_4^2 - 4A_2A_6\right)^{\frac{1}{2}}\right]/2A_6 \equiv M_\pm^2.$$

The $+$ sign gives the minima and the $-$ sign the maxima.

T_0 is determined by $A(M) = 0$ having a double root at $M = \pm M_+$ (note that A_0 is set to zero so that $A(0) = 0$ is the minimum for $T > T_0$). The solution is

$$A_2 = \frac{3}{16}\frac{A_4^2}{A_6}, \tag{34}$$

and at the transition

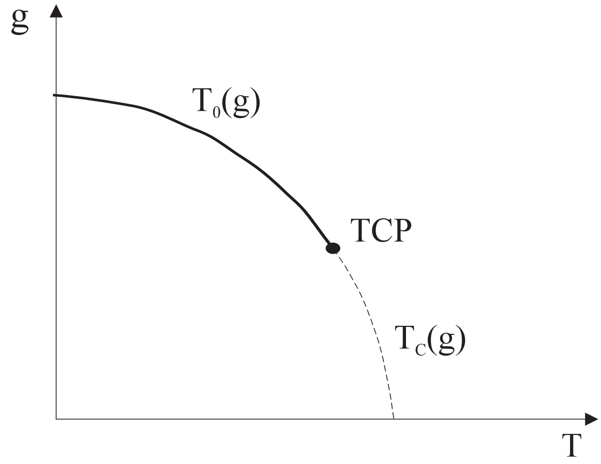
$$\left(\Delta M^2\right) = M_+^2 = -\frac{3}{4}\frac{A_4}{A_6}$$

Thus the point $T = T_c, A_4 = 0$ separates the first-order line from the second-order line: this is a **tricritical point**. To see the tricritical point these two parameters have to take these special values and this requires tuning two external fields in the phase diagram, although in the most general case, when odd powers of M are included, up to four external fields must be tuned.

Using Eq. (34) we see that $A_2 = a_2(T_0(g) - T_c(g)) = \frac{3}{16}\frac{A_4^2}{A_6} \geq 0$ when $A_4(g) \leq 0$, and so we find

$$\begin{array}{llll} T = T_0(g) & > T_c(g): & \text{First order phase transition,} & A_4 < 0 \\ T = T_0(g) & = T_c(g): & \text{Tricritical (TCP) phase transition,} & A_4 = 0 \\ T = T_c(g): & & \text{Continous phase transition,} & A_4 > 0 \end{array}$$

where the space of external fields is denoted by T and g (e.g., g can be identified with a chemical potential controlling the relative abundances in a two component system). In terms of these variables the phase diagram has the form:



We get different critical exponents for a TCP. As an example we calculate δ: at $T = T_{TCP}$ $M \sim h^{1/\delta}$. The result can be derived immediately from Landau theory. When $T = T_{TCP}$ **both** A_2 and A_4 are zero and so we can write

$$A = -hM + A_6 M^6 .$$

Thus we find that

$$\frac{\partial A}{\partial M} = 0 \quad \Rightarrow \quad M \sim h^{\frac{1}{5}} .$$

A summary of the critical indices is

	$A \sim \lvert t\rvert^{2-\alpha}$ α	$M \sim (-t)^{\beta}$ β	$\chi \sim \lvert t\rvert^{-\gamma}$ γ	$M \sim h^{\frac{1}{\delta}}$ δ
CP	0	$\frac{1}{2}$	1	3
TCP	$\frac{1}{2}$	$\frac{1}{4}$	1	5

7 Mean field theory

Mean field theory works in the limit $D \to \infty$. However, as we asserted earlier and shall see later, the predictions for critical exponents are correct for $D > D_c$ where D_c is the **critical** dimension. Note that D_c is different for different kinds of critical point. E.g., $D_c = 4$ for and ordinary CP but $D_c = 3$ for a TCP.

We use the Ising model as the example to explain the ideas. In D dimensions there are $2D \equiv q$ nearest neighbours to each site. We have

$$H \;=\; -\,J \sum_{\mathbf{n},\mu} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\mu} \;-\; h \sum_{\mathbf{n}} \sigma_{\mathbf{n}} \,. \tag{35}$$

We write the identity (see Cardy)

$$\sigma_{\mathbf{n}}\sigma_{\mathbf{n}'} \;=\; (M + (\sigma_{\mathbf{n}} - M))\,(M + (\sigma_{\mathbf{n}'} - M)) \tag{36}$$

and define $\delta\sigma_{\mathbf{n}} = (\sigma_{\mathbf{n}} - M)$. Here M is the magnetization per spin. We expand this expression and get

$$\sigma_{\mathbf{n}}\sigma_{\mathbf{n}'} \;=\; -\,M^2 \;+\; M\,(\sigma_{\mathbf{n}} + \sigma_{\mathbf{n}'}) \;+\; \delta\sigma_{\mathbf{n}}\,\delta\sigma_{\mathbf{n}'}\,. \tag{37}$$

The **mean field approximation** consists in ignoring the last term which represents the interaction between sites of the fluctuations $\delta\sigma$ of the spins about their mean value. What this really means is that we are asserting that the rôle of fluctuations is not important for the quantities we want to calculate. For example, it may be that their whole effect is just to modify the value of J to some effective value or they modify overall coefficients, but otherwise the theory gives the same answers for phase structure etc. without explicitly including them.

This assumption is related to the central limit theorem. As $D \to \infty$ we expect that the magnitude of

$$\langle\, \delta\sigma_0\,(\tfrac{1}{2D}\sum_{\mathbf{n}}\delta\sigma_{\mathbf{n}})\,\rangle \;\sim\; 1/\sqrt{D}\,, \qquad \mathbf{n} \;\in\; 2D \text{ nearest neighbours of } 0\,,$$

This measures the **average** interaction of fluctuations between a spin and its nearest neighbours. Thus, the effect of fluctuations becomes negligible for large enough D. How big D must be and how the assumption breaks down is the interesting question.

Then the mean field Hamiltonian H is

$$H \;=\; \frac{1}{2}qJNM^2 \;-\; (qJM + h)\sum_{\mathbf{n}}\sigma_{\mathbf{n}} \tag{38}$$

and

$$\begin{aligned} Z \;&=\; e^{-\frac{1}{2}\beta qJNM^2}\sum_{\sigma} e^{\beta(qJM+h)\sum_{\mathbf{n}}\sigma_{\mathbf{n}}} \\ &=\; e^{-\frac{1}{2}\beta qJNM^2}\,\left[2\cosh\beta(qJM+h)\right]^N\,. \end{aligned} \tag{39}$$

Note that our discrete Ising spin has been replaced by a continous variable or field: the magnetization M. This is a hint at why many different models are in the same universality class.

The magnetization is given by

$$\begin{aligned} M \;&=\; \langle\sigma_{\mathbf{0}}\rangle \;=\; \frac{\displaystyle\sum_{\sigma=\pm1}\sigma\, e^{\beta(qJM+h)\sigma}}{\displaystyle\sum_{\sigma=\pm1} e^{\beta(qJM+h)\sigma}} \\ &=\; \tanh\beta(qJM+h)\,. \end{aligned} \tag{40}$$

So

$$\begin{aligned} A \;&=\; F/N \;=\; -\,kT\log\mathcal{Z} \;= \\ &=\; -kT\log\left[2\cosh\beta(qJM+h)\right] \;+\; \frac{1}{2}qJM^2\,. \end{aligned} \tag{41}$$

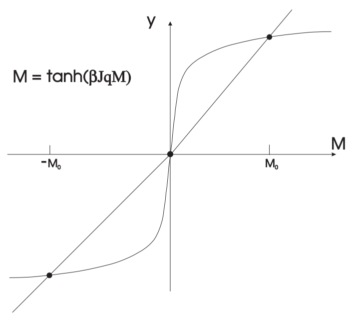
The result for M above (40) then also arises from minimizing the free energy A:

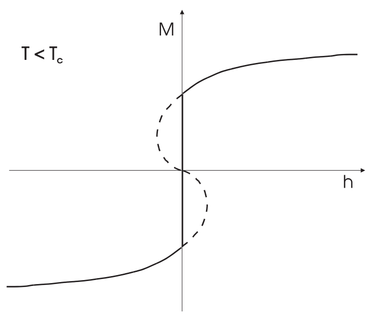
$$\left(\frac{\partial A}{\partial M}\right)_{T,h} = 0 \quad \Rightarrow \quad M = \tanh \beta(qJM + h)\,. \tag{42}$$

This is the **equation of state**. For small M, h (i.e., ignoring non-linear terms in h and a few other non-essential terms) we can expand A:

$$\begin{aligned} A &= \\ &-kT\log 2 - \frac{1}{\beta}\log\left[1 + \beta^2(qJM+h)^2/2 + \beta^4(qJM+h)^4/24 + \ldots\right] + \frac{1}{2}qJM^2 \\ &= A_0(T) - (\beta qJ)\,hM + \frac{1}{2}qJ(1-\beta qJ)\,M^2 + \frac{1}{12}qJ(\beta qJ)^3\,M^4 + \ldots\,, \end{aligned} \tag{43}$$

where A_0 absorbs terms independent of M, h. The equation of state is obtained as stated by minimizing A as in (42). This can be clearly be interpreted in the spirit of Landau-Ginsburg theory and we see that there is a second order phase transition at $T = T_c$ where $\beta_c qJ = 1$ or $T_c = qJ/k$. The diagrammatic solution to the equation of state is shown in the figures below for $T < T_c$. There is always a solution $M = 0$ but for $T < T_c$ this corresponds to a **local maximum** in $A(M)$ and therefore does not correspond to an equilibrium state. We have seen how this occurs in Landau theory and in this explicit example we can verify it by expanding A in M for $T \sim T_c$ and small h. In the second graph the smooth curve is the solution from the equation of state for $M(h)$, $T < T_c$ but the correct function describing equilibrium is the solid curve given by the Maxwell construction. Points on the vertical part at $h = 0$ correspond to different mixtures of domains whose average gives the value of M. The reason we do not get this from the statistical mechanics directly is that we have explicitly dealt with homogeneous or translation invariant systems which excludes this structure. Metastable states can be reached where $\chi = \partial M/\partial h > 0$. This gives rise to hysteresis effects.





To examine these results and to obtain leading order results we can set $\beta qJ = 1$ in most places in (43) and use $T \sim T_c$ to find

$$A \sim A_0 + hM + \frac{1}{2}k(T-T_c)\,M^2 + \frac{1}{12}kT_c\,M^4 + \ldots\,. \tag{44}$$

This makes the connection with Landau-Ginsburg theory very clear. This is the same expression as in LG theory, equation (31). For example, for $T < T_c$ $M = 0$ is a maximum of A. However, we have suppressed all but the important T dependence and so we should only use this expression where appropriate.

8 The scaling hypothesis

In Landau theory we have

$$A \;=\; A_0 - hM + \frac{1}{2}A_2M^2 + \frac{1}{4}A_4M^4 + \ldots ,$$

and

$$\begin{aligned} \frac{dA}{dM} &= 0 \implies \\ -h + A_2M + A_4M^3 &= 0 , \end{aligned}$$

where A is the free energy per unit volume. M is, for example, the magnetization per unit volume in an Ising system or the deviation $\rho^* = \rho - \rho_c$ of the density ρ from its critical value ρ_c in a liquid.

We have seen how this leads to predictions for critical exponents. However, these prediction do not always agree with experimentally observed values even though the topological structure of the phase diagram can be well described by the Landau approach. Because power-law behaviour is still the right form to use in fitting experiment it suggests an approach based on a form of dimensional analysis or on a scaling formulation.

Suppose we scale units by a factor "k". Then, since, A is the free energy per unit volume

$$A \;\rightarrow\; A' \;= k^{-D}A \;.$$

[This means that if the old unit is a and the new unit a' then $a' = k^{-1}a$. E.g., consider a length expressed in the two kinds of units. We must have

$$\begin{aligned} L'a' &= La \;\Rightarrow\; L' = kL \\ V'{a'}^D &= Va^D \;\Rightarrow\; V' = k^DV \end{aligned}$$

]

Denote the dimension of A by $[A]$ so that $[A] = -D$. Suppose $[M] = D_M$ but a-priori we do not know the value of D_M. Note that it is not a general property that we can assign a dimension to M. It will only be a useful concept near a continuous transition. Of course, we can always use standard or "engineering" dimensional analysis but here we are proposing something more general. Then

$$[A_n] \;=\; -\,(D + nD_M) \;.$$

The phase transition is caused by A_2 changing sign and we assume that $A_4 > 0$ (however, see TCP) and so the series can be truncated after M^4. We use the idea that A is an homogeneous function and so we construct dimensionless ratios made of h ($\equiv A_1$), A_2, A_4:

$$[h] \;=\; -\,(D + D_M), \;\; [A_2] \;=\; -\,(D + 2D_M), \;\; [A_4] \;=\; -\,(D + 4D_M) \;.$$

Consider $hA_2^{p_2}A_4^{p_4}$ and we find, for D_M and D unrelated,

$$1 + p_2 + p_4 \;=\; 0 \quad 1 + 2p_2 + 4p_4 \;=\; 0 \;\Longrightarrow\; p_2 \;=\; -\frac{3}{2}\,, \;\; p_4 \;=\; \frac{1}{2}\,.$$

The only dimensionless term is thus

$$\frac{hA_4^{1/2}}{|A_2|^{3/2}}\,.$$

A_4 is not particularly interesting: the denominator $|A_2|$ is the interesting part. It is natural to use $|A_2|$ here and not A_2 since as A_2 vanishes we must ensure continuity of expressions. In other words we want continuity of A at $T = T_c$ and will express the behaviour for $T > T_c, T < T_c$ as separate functions of $|t|$, $t = (T - T_c)$.

To carry the dimension of A we need to construct a variable with dimension $-D$.

(a) $h = 0$. Find

$$\frac{|A_2|^2}{A_4} \;\sim\; |t|^2\,.$$

(b) $t = 0 \Longrightarrow A_2 = 0$. Find

$$\frac{h^{4/3}}{A_4^{1/3}}\,.$$

So, using (a), we can write the singular part of A in its scaling form appropriate for small h

$$A \;=\; a|t|^2 f_{\substack{>\\<}}\left(b\frac{h}{|t|^{3/2}}\right)\,. \tag{45}$$

The functions $f_{\substack{>\\<}}$ apply for $t > 0$ and $t < 0$, respectively, with $f_<(0) = f_>(0)$. Note that the exponents 2 and 3/2 are universal but a and b are not. The theory will describe e.g., both the ferromagnet and H_20.

Then

$$M \;=\; \left(\frac{\partial A}{\partial h}\right)_{h\to 0+} \;\sim\; |t|^2 \frac{1}{|t|^{3/2}} f'_{\substack{>\\<}}(0+) \quad\Longrightarrow$$

$$M \;\sim\; |t|^{1/2} \qquad t < 0 \quad f'_<(0+) \;>\; 0$$

$$M \;=\; 0 \qquad t > 0 \quad f'_>(0+) \;=\; 0$$

Note that $f'_<(0+) \;=\; -f'_<(0-)$ by symmetry. This is evident from our previous discussion of Landau theory.

The susceptibility χ is then

$$\chi \;=\; \left(\frac{\partial M}{\partial h}\right) \;\sim\; |t|^{-1} A_{\substack{>\\<}} \quad\Longrightarrow\quad \gamma \;=\; 1\,.$$

$$A_> \;=\; ab^2 f''_>(0)\,, \quad A_< \;=\; ab^2 f''_<(0)\,.$$

We have the specific heat C given by

$$C \;=\; T\left(\frac{\partial^2 A}{\partial T^2}\right)\Bigg|_{h=0} \; .$$

From (45) find $C \sim t^0 \implies \alpha = 0$

Both $A_>, A_<$ are non-zero and positive. Clearly, the ratio of $A_>$ to $A_<$ does not depend on the non-universal parameter ab^2. Thus, in addition to the critical exponents, we expect $A_>/A_<$ to be universal if the functions $f_{\gtrless}$ are universal. This is true in Landau theory and we will see that it is true generally.

From (b) above we can also write the singular part of A appropriate for small t and non-zero (positive) h as

$$A \;=\; c h^{4/3}\, g_{\gtrless}\left(\frac{1}{b}\frac{|t|^{3/2}}{h}\right) \; .$$

A similar analysis to that above pertains.

In this way we reproduce all the critical exponents that we derived before. The scaling generalizes the approach to accommodate the observed values within this general framework.

The hypothesis is the postulate that we can write

$$A \;=\; |t|^{2-\alpha}\, f_{\gtrless}\left(\frac{h}{|t|^{\Delta}}\right) \; , \tag{46}$$

from which we associate α with the scaling of the free energy. This form is appropriate when t is non-zero. Clearly, from above the simple scaling argument gives $A \sim |t|^2 \Rightarrow \alpha = 0$. However, we now allow α, Δ to be free parameters.

Following our earlier steps we get

$$\begin{aligned}
M &\sim |t|^{\beta} \quad t < 0\,, & \beta &= 2 - \alpha - \Delta \\
\chi &\sim |t|^{-\gamma} & -\gamma &= 2 - \alpha - 2\Delta \\
\implies \alpha &+ 2\beta + \gamma \;=\; 2 & &\textbf{scaling relation}
\end{aligned}$$

This is to be compared with the Rushbrooke inequality deduce from thermodynamics: $\alpha + 2\beta + \gamma \geq 2$ (problem sheet).

Other exponents are calculated with $t = 0$. Take $h > 0$ and then we can rewrite

$$\begin{aligned}
M \;&=\; (\partial A/\partial h) \;\sim\; |t|^{\beta} f'_<\left(\frac{h}{|t|^{\Delta}}\right) \\
&=\; h^{\beta/\Delta}\, \underbrace{\left(\frac{|t|^{\Delta}}{h}\right)^{\beta/\Delta} f'_<\left(\frac{h}{|t|^{\Delta}}\right)}_{\phi\left(\frac{|t|^{\Delta}}{h}\right)} \; .
\end{aligned} \tag{47}$$

We **assume** that $\lim_{x\to 0}\,\phi(x)$ is finite and $\neq 0$. Then for $t=0$

$$\begin{aligned} M &\sim h^{1/\delta}\phi(0) \quad \Longrightarrow \\ 1/\delta &= \frac{\beta}{\Delta} = \frac{(2-\alpha-\Delta)}{\Delta} \quad \Longrightarrow \\ \beta\,\delta &= \Delta = \beta + \gamma \qquad \textbf{scaling relation} \end{aligned}$$

On the assumption that the functions defined have suitable limits we have derived scaling relationships between critical indices. This is because they are ultimately dependent only on the two independent exponents α and Δ. This has been achieved by invoking a generalized scaling theory which is dimensional analysis but with dimensions which must be predicted by theory. We also see how **universality** for exponents and amplitude ratios (such as $A_>/A_<$) arise. This is because very different theories can have order parameters with the same properties. For example, magnetization M and density $\rho^* = (\rho - \rho_c)$. The Landau analysis is the same for all such theories which thus fall into **universality classes**. This will be justified by the Renormalization Group (RG) analysis.

We shall also see that there is an exponent ν associated with the divergence of the correlation length ξ defined by

$$\xi \sim |t|^{-\nu} \qquad t \to 0\,.$$

9 Critical properties of the 1D Ising model

Except in special circumstances there are no phase transitions at non-zero T in one dimension. The entropy S is largest when the system is disordered and its contribution to $F = U - TS$ dominates the internal energy term U except at $T = 0$. Hence, the minimum of F always corresponds to a disordered state: a state usually associated with high T in higher dimensional models. In the 1D Ising model there is critical behaviour as $T \to 0 \Rightarrow T_c = 0$. We can see this by computing the correlation length for $h = 0$ and low T:

$$\begin{aligned} \xi^{-1} &= \log(\lambda_+/\lambda_-) = \log\coth\beta J \\ &\sim 2\,\exp(-2J/kT) \quad T \to 0 \\ \Rightarrow & \\ \xi &\sim \frac{1}{2}\exp(2J/kT) \quad T \to 0\,. \end{aligned} \tag{48}$$

Thus ξ diverges as $T \to 0$ and this signifies a continuous transition at $T_c = 0$. The approach to the transition is actually $T/J \to \infty,\ h/T \to 0$, which includes $J \to \infty$ at fixed $T > 0$. Because of this we use $t = \exp(-4J/kT)$ to measure the departure from criticality rather than $(T - T_c)/T_c$ (there is a abritary nature to this choice, see later). To encode the magnetic field dependence we use $B = h/kT$ as defined before. Then from section 4 equation (25) we find for small t and B

$$M \sim \frac{B}{\sqrt{t+B^2}} = \frac{1}{\sqrt{(t/B^2)+1}} .$$

Compare with the form for M given by the scaling hypothesis (47) in the previous section

$$M \sim B^{1/\delta}\,\phi\left(\frac{t^\Delta}{B}\right) ,$$

and we read off that $\delta = \infty$ and $\beta\delta = \beta + \gamma = \Delta = 1/2$. This gives $\beta = 0, \gamma = 1/2$, and **if** the scaling relation $\alpha + 2\beta + \gamma = 2$ holds true we find $\alpha = 3/2$. Also we identify

$$\phi(x) = 1/\sqrt{x^2+1} .$$

The exponent ν is defined by $\xi \sim t^{-\nu}$ and from above we have

$$\xi \sim t^{-1/2} \quad \Rightarrow \quad \nu = 1/2 .$$

The exponent η fleetingly referred to before is defined by

$$G(r) \sim \frac{1}{r^{D-2+\eta}} \qquad r \ll \xi ,$$

and we have from section (4) equation (30) that $G(r) \sim \sin^2 2\phi$ in this limit and hence $\eta = 1$.

In fact, the definition of t is rather arbitrary and any power of t will do equally well. This, of course, affects some of the exponents and so the better way to express these results is

$$2 - \alpha = \gamma = \nu , \quad \beta = 0,\ \delta = \infty,\ \eta = 1 .$$

Remember the relation for α depends on assuming $\alpha + 2\beta + \gamma = 2$ but because the definition of α is tricky in this case, it is not obvious that we can invoke this relation. We can calculate the free energy directly from the exact solution and see what is going on (see problem sheet 2). This is a pathology of the 1D Ising model which is unusual in this respect.

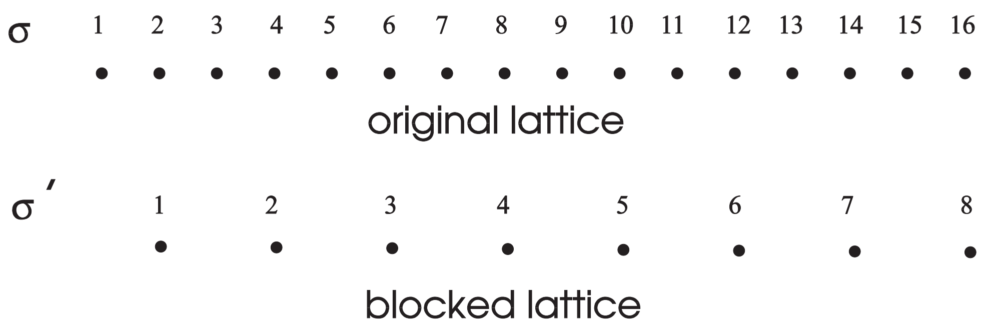
10 The blocking transformation

The critical properties of a system are governed by the behaviour of degrees of freedom which vary typically on the scale of the correlation length, ξ. Since, ξ diverges at the critical point we are interested in the dynamics of modes whose wavelengths are much larger than a lattice spacing and we would like to find an **effective** theory or Hamiltonian which describes the large scale properties of the system but which is, itself, defined only in terms of these long-wavelength modes. Effectively, we want to "integrate" out the short-wavelength modes and come up with a new theory with fewer degrees of freedom but which has the same critical properties as the original system.

We do something like this all the time. Chemists deal with collections of atoms for which they have a way of describing their interactions and bonding properties. When building a large molecule they generally do not write down Schödinger's equation for all the constituent electrons and nuclei and solve from first principles. Instead, they have an effective description in which the atoms are entities which interact on a scale larger than the internal atomic scales. Their description is not so elegant as the original Schödinger equation but it is practical and concentrates on addressing the physics relevant to their problem. "Integrating" out the small wavelength modes gave the atomic bound states in the first place. Again, we treat atomic nuclei as particles with charge and if we want to study atomic physics we do not feel it necessary to solve the nuclear physics of the nucleus at the same time. We loose information (such as which isotopes are stable etc.) but we do not mind. Of course, if the chemist heats up the molecules a lot, the atoms will eventually ionize and the electrons and nuclei will become separate particles in a plasma. The atomic – long-range – description is no longer any good.

In statistical systems we "thin" the degrees of freedom using a blocking strategy. This can sometimes be exact but generally is approximate. The hope is that we retain the correct critical properties and phase structure in the new theory. A bad blocking scheme will eliminate degrees of freedom that are necessary for critical behaviour and the scheme will fail.

We demonstrate the idea in the 1D Ising model where it is exact. We consider a system with periodic boundary conditions with couplings J, h and comprised of $N = 2^n$ ising spins. We demand that the partition function for this model is reproduced by a similarly defined model with $J', h', N' = 2^{n-1}$. We choose the second model to be defined on the spins with **even** labels n only and perform the sum over spins with n **odd**. The diagram shows the relation between the two models.



We then can define the blocking transformation

$$\exp(-\beta H(J', h', N', \sigma') - \beta N'C') = \sum_{\sigma} \left(\prod_{p=1}^{N'} \delta_{\sigma'_p, \sigma_{2p}}\right) \exp(-\beta H(J, h, N, \sigma) - \beta NC) , \tag{49}$$

with

$$H(J, h, N, \sigma) = -J \sum_n \sigma_n \sigma_{n+1} - h \sum_n \sigma_n \quad n = 1, 2, \ldots, N \ , \ \sigma_N = \sigma_0 \ . \tag{50}$$

We have allowed for an additive constant NC in the partition function which we are always free to do since it just trivially adds to the free energy. We shall see why it is necessary below. (Note Yeomans introduces a similar C but mine has the opposite sign to hers – I prefer that C **adds** to the free energy.)

The lattice spacing in the blocked model is 2 times the original model: a new **block spin** σ'_p has replaced blocks of two original spins (σ_{p-1}, σ_p) . The blocking function that defines how this is done is denoted $T(\sigma', \sigma)$ and here

$$T(\sigma', \sigma) \;=\; \prod_{p=1}^{N'} \delta_{\sigma'_p, \sigma_{2p}} \; . \tag{51}$$

T satisfies the condition

$$\sum_{\sigma'} T(\sigma', \sigma) \;=\; 1 \; . \tag{52}$$

This ensures that the blocked model has the original partition function $\mathcal{Z}(J, h, C, N)$:

$$\begin{aligned}
\mathcal{Z}(J', h', C', N') =& \sum_{\sigma'} \exp(-\beta H(J', h', N', \sigma') - \beta N'C') \\
=& \sum_{\sigma'} \sum_{\sigma} T(\sigma', \sigma) \; \exp(-\beta H(J, h, N, \sigma) - \beta NC) \\
=& \sum_{\sigma} \exp(-\beta H(J, h, N, \sigma) - \beta NC) \\
=& \; \mathcal{Z}(J, h, C, N)
\end{aligned}$$

The volumes of the two systems are the same: they describe the same system. We define the blocking factor $b = N/N'$, and the lattice spacing has been scaled by $k = b^{-1}$ (remember, by definition $a' = k^{-1}a$ and here $a' = ba$ so that lengths in **lattice units** scale as $L' = kL$). In the example here $b = 2$.

This particular form of blocking is called decimation and generally does not work well but in our soluble model it is exact.

Another blocking strategy might for example be the majority rule where we take blocks with an odd number $2q + 1$ of spins and then $b = 2q + 1$. The block spin is then given by

$$\sigma'_r \;=\; \frac{\sum_{p=1}^{b} \sigma_{n+p}}{\left| \sum_{p=1}^{b} \sigma_{n+p} \right|} \qquad n = (r-1)b + 1 \quad r = 1, 2, \ldots \; .$$

That is, σ' is the majority sign of the block.

In the example we are considering we note that from (49) that the transfer matrix of the blocked model is the square of that of the original model. So

$$\begin{aligned}
W'_{\sigma'_p\, \sigma'_{p+1}} \;&=\; \sum_{\sigma_{2p+1}} \delta_{\sigma'_p, \sigma_{2p}} \delta_{\sigma'_{p+1}, \sigma_{2p+2}} W_{\sigma_{2p}\, \sigma_{2p+1}} W_{\sigma_{2p+1}\, \sigma_{2p+2}} \\
\Rightarrow& \\
W' \;&=\; W^2
\end{aligned} \tag{53}$$

It is convenient to define $z = e^{\beta J}, \mu = e^{\beta h}, c = e^{-\beta C}$. Then from section 4 we have

$$W = c \begin{pmatrix} \mu z & z^{-1} \\ z^{-1} & \mu^{-1} z \end{pmatrix} \tag{54}$$

and similarly for W'. We then have the relation

$$c' \begin{pmatrix} \mu' z' & z'^{-1} \\ z'^{-1} & \mu'^{-1} z' \end{pmatrix} = c^2 \begin{pmatrix} \mu z & z^{-1} \\ z^{-1} & \mu^{-1} z \end{pmatrix} \begin{pmatrix} \mu z & z^{-1} \\ z^{-1} & \mu^{-1} z \end{pmatrix} . \tag{55}$$

These are three equations in three unknowns and so there is a solution. Without the introduction of the additive constant C in H this would not be the case. We find

$$c' z' \mu' = c^2 (z^2 \mu^2 + 1/z^2) \tag{56}$$
$$c'/z' = c^2 (\mu + 1/\mu) \tag{57}$$
$$c' z'/\mu' = c^2 (z^2/\mu^2 + 1/z^2) \tag{58}$$

The results are best expressed in terms of

$$x = e^{-4\beta J} = 1/z^4 , \quad y = e^{-2\beta h} = 1/\mu^2 , \quad w = e^{4\beta C} = 1/c^4 .$$

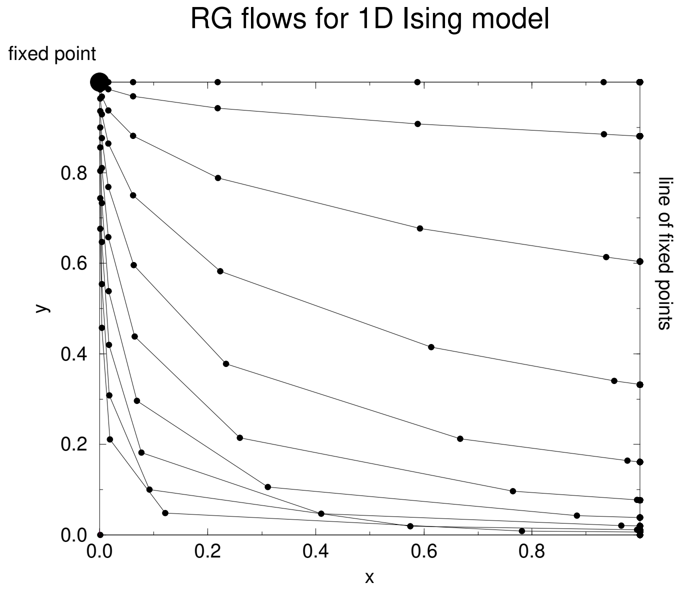
We then find the recursion relations

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)} \tag{59}$$
$$y' = \frac{y(x+y)}{(1+xy)} \tag{60}$$
$$w' = \frac{w^2 x y^2}{(1+y)^2 (x+y)(1+xy)} . \tag{61}$$

These equations are called **Renormalization Group equations**. They relate the coupling constants of an original model to those of a similar model which has degrees of freedom reduced by a factor $b = N/N'$ ($b = 2$ here) but which gives rise to the **same** large-scale thermodynamic behaviour. This is clear because the thermodynamic quantities in which we are interested are macroscopic degrees of freedom which are determined from the partition function by differentiating with respect to J, h.

Notes

(1) The fields J, h couple to long range quantities since they are constants. To obtain the connected pair correlation function $G(r)$ we introduced a varying field h_n but now we consider this field to vary very slowly on the range of one lattice spacing a. We are ultimately interested only in $G(r)$ for $r \gg a$ and so the blocked model retains all the required information since only short wavelength degrees of freedom have been removed. Their contribution is encoded in the changed coupling constants.



(2) The RG equations can be iterated and this gives rise to RG trajectories in (x, y) space. All models lying on a given trajectory give rise to the **same** long-range physics. The graph below shows the structure in this case when $h \geq 0$.

There is a line of fixed points at $x = 1\ \forall\ y$. This corresponds to $T/J = \infty$ for which the model is maximally disordered. This includes the theories where $J = 0$ or $T = \infty$. We expect this since for $T > 0$ the system is in the high temperature or disordered phase and so under blocking any fixed point must correspond to disorder. The point at $x = 0, y = 1$ is a fixed point which has $T = 0, h = 0$ and corresponds to critical behaviour.

(3) The correlation length in physical units is unchanged by the blocking. However, in the context of a given model, ξ is measured in units of the lattice spacing a. After p iterations of the blocking we then must have

$$a \to b^p a \quad \Rightarrow \quad \xi \to \xi_p = b^{-p}\xi \ .$$

At a fixed point of the RG equations the correlation length must be unchanged: $\xi_p = \xi$. This is only possible for $\xi = 0$ or $\xi = \infty$.

A fixed point with	$\xi = 0$	is a **trivial** fixed point
A fixed point with	$\xi = \infty$	is a **non-trivial** fixed point and corresponds to a continuous phase transition

See the figure above and note (2). We have seen that any model except at $\beta J = \infty, \beta h = 0$ has a finite ξ and so we expect all these models to flow to a trivial fixed point. Note that $\xi \to \infty$ as $T \to 0$ for $J \neq 0, h = 0$ which is where we expected ξ to diverge.

(4) Near a fixed point the RG equations can be linearized.

(i) $x = 1 - \epsilon, \;\; \forall\, y$:

$$\epsilon' \;=\; \epsilon^2 \frac{y}{(1+y)^2}\;, \qquad y' \;=\; y(1 - \epsilon\frac{1-y}{1+y})$$

This gives the line of trivial fixed points defined by $\epsilon = 0 \;\; \forall\, y$. The first equation has no linear term.

(ii) $x = \epsilon, \;\; y = 1 - \rho$:

$$\epsilon' \;=\; 4\epsilon + \ldots\;, \qquad \rho' \;=\; 2\rho + \ldots\;.$$

This gives the behaviour in the neighbourhood of the fixed point corresponding to the continuous phase transition. The linear transformation is already diagonal with eigenvalues 4 and 2, respectively. It is useful to write these eigenvalues as powers of b, i.e. we write ($b = 2$)

$$\epsilon' \;=\; b^{\lambda_t}\epsilon\;, \qquad \rho' \;=\; b^{\lambda_h}\rho\;, \qquad \text{with} \qquad \lambda_t = 2,\; \lambda_h = 1\;.$$

We will see that λ_t and λ_h play the rôle of dimensions associated with x (and hence t) and y (and hence h).

(5) $C = \frac{1}{4}kT\log(w)$ accumulates the free energy contributed by the degrees of freedom which are summed over. It appears as an additive constant in the exponent in $\mathcal{Z}$. We consider the free energy **per spin** and note that it is composed of two distinct parts:

$$\begin{aligned} F(J,h,C) \;&=\; f(J,h) \;+\; C \\ f(J,h) \;&=\; -\frac{kT}{N}\log\mathcal{Z}(J,h,C=0,N)\;. \end{aligned} \tag{62}$$

Then one iteration of the blocking gives

$$\begin{aligned} &\exp(-\beta N F(J,h,C)) \;=\; \exp(-\beta N' F(J',h',C')) \\ \Rightarrow& \\ &F(J,h,C) \;=\; \frac{N'}{N}F(J',h',C') \;=\; b^{-1}F(J',h',C') \end{aligned} \tag{63}$$

After p iterations of blocking denote the coupling constants obtained by $u_p = (J_p, h_p)$ and the free energy by $F(u_p, C_p)$. Then we have

$$\begin{aligned} &F(u_0,C_0) \;=\; b^{-p}F(u_p,C_p) \\ \Rightarrow& \\ &f(u_0) \;=\; b^{-p}f(u_p) \;+\; b^{-p}C_p \;-\; C_0. \end{aligned} \tag{64}$$

We see that C_p accumulates the contribution to the free energy from the spins which have been summed over. From the RG equation (61) for w (take logs of both sides) we see that

$$C_p \;=\; b\,C_{p-1} \;+\; b\,g(u_{p-1}) \tag{65}$$

$$\Rightarrow$$

$$b^{-p}C_p \;=\; \sum_{j=0}^{p-1} b^{-j}g(u_j) + C_0\;, \qquad p > 0.$$

The RG equations thus **define** the function $g(u)$ by (65) above.

We then have an important equation relating the free energy per spin of the original model to that of the blocked model.

$$f(u_0) \;=\; \underbrace{b^{-p}f(u_p)}_{\text{singular part}} \;+\; \underbrace{\sum_{j=0}^{p-1} b^{-j}g(u_j)}_{\text{inhomog. part}} \;. \tag{66}$$

with $g(u)$ defined from the RG equation (61). The RHS contains a so-called "singular" part and an inhomogeneous part: the free energy does not transform homogeneously. Most authors throw away the inhomogeneous part by various arguments. However, (as Ma remarks) this is not correct but can be shown to give the right answers and allows us to concentrate on the homogeneous transformation of the singular part. I will remark on this later.

Near to the fixed point $(0,1)$ we can expand the RG equation for C $(x=\epsilon,\; y=1-\rho)$:

$$\begin{aligned} & w' \;=\; \frac{1}{4}w^2\,\epsilon\,((1-2\epsilon)\;+\;\ldots) \\ \Rightarrow \\ & C' \;=\; 2C\;+\;kT(\frac{1}{4}\log\epsilon-\frac{1}{2}\log 2-\frac{1}{2}\epsilon\;+\;\ldots) \\ \Rightarrow \\ & g(u) \;=\; kT(\frac{1}{8}\log\epsilon-\frac{1}{4}\epsilon-\frac{1}{4}\log 2\;+\;\ldots)\;, \end{aligned} \tag{67}$$

where $\ldots$ stands for terms higher order in ϵ, ρ. Note that in this approximation $g(u)$ does **not** depend linearly on h since there is no linear term in ρ.

In this example we have demonstrated all of the technology of the RG and blocking. To derive the phenomenon of scaling and to see how critical indices arise we give a general description of the RG in the next section but draw heavily on the understanding gained from this example.

11 The Real Space Renormalization Group

Consider a system defined in D dimensions on a lattice of spacing a with N sites and with a spin, or field, σ_r on the r-th site. Note now tht σ can be discrete or continuous. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$, e.g., nearest neighbour, next nearest neighbour, multiple groups etc. The general Hamiltonian is

$$H(\mathbf{u},\sigma) \;=\; \sum_i u_i O_i(\{\sigma\})\;, \tag{68}$$

where the u_i are the associated coupling constants and $\mathbf{u}=(u_1,u_2,\ldots)$. The partition function is then

$$\mathcal{Z}(\mathbf{u},C,N) \;=\; \sum_\sigma \exp(-\beta H(\mathbf{u},\sigma)-\beta NC)\;,$$

where C again will accumulate the free energy contribution from summed-over spins.

The RG transformation is defined in terms of a blocking kernel $T(\sigma', \sigma)$

$$\exp(-\beta H(\mathbf{u}', \sigma') - \beta N'C') \;=\; \sum_{\sigma} T(\sigma', \sigma) \exp(-\beta H(\mathbf{u}, \sigma) - \beta NC) \;.$$

We have seen an example of a blocking kernel in the previous section. The transformation corresponds to summing over a subset of spins to give a model whose Hamiltonian has the same form as the original one (68) but with different couplings. The feature of the transformation is that the new, or blocked, model is defined on a lattice with increased lattice spacing a' and reduced number of sites N'.

$$a \;\to\; a' \;=\; ba \qquad b^D \;=\; N/N' \,.$$

In earlier discussion of scaling in section 8 we defined k by $a' = k^{-1}a \;\;\Rightarrow\;\;$ identify $k = b^{-1}$ here.

E.g., in the $1D$ Ising model $b = 2$. We denote the RG transformation of the couplings by $\mathbf{R}$, so that under blocking

$$\mathbf{u}_p \;\to\; \mathbf{u}_{p+1} \;=\; \mathbf{R}(\mathbf{u}_p) \,.$$

We have

$$\mathcal{Z}(\mathbf{u}_p, C_p) \;=\; \mathcal{Z}(\mathbf{u}_{p-1}, C_{p-1}) \;\; \forall p \;>\; 0 \;.$$

Using results from the previous section we have

$$\begin{aligned} F(\mathbf{u}_0, C_0) \;&=\; b^{-pD} F(\mathbf{u}_p, C_p) \\ \Rightarrow \qquad & \\ f(\mathbf{u}_0) \;&=\; b^{-pD} f(\mathbf{u}_p) \;+\; b^{-pD} C_p \;-\; C_0, \end{aligned} \tag{69}$$

and hence that

$$f(\mathbf{u}_0) \;=\; \underbrace{b^{-pD} f(\mathbf{u}_p)}_{\text{singular part}} \;+\; \underbrace{\sum_{j=0}^{p-1} b^{-jD} g(\mathbf{u}_j)}_{\text{inhomog. part}} \;. \tag{70}$$

Again, the inhomogeneous part is the contribution from the degrees of freedom that have been summed over.

Distances measured in terms of the lattice spacing as a unit also scale with b

$$\mathbf{r} \;\to\; \mathbf{r}' \;=\; b^{-1}\mathbf{r}$$

and in particular this applies to the correlation length $\xi \to \xi' = b^{-1}\xi$. Remember, the original scale parameter is $k = b^{-1}$ and so we see that as expected $[F] = -D, [\mathbf{r}] = 1, [\xi] = 1$.

We might expect that the pair correlation function behaves so that

$$G(\mathbf{r}, \mathbf{u}) \;=\; G(b^{-1}\mathbf{r}, \mathbf{u}') \,,$$

certainly for $r \gg a$. However, we must also allow for a field rescaling factor where appropriate. To understand how this works we make the following points:

(1) σ is a dummy variable of summation if it is discrete, or a dummy variable of integration if a continuous field. Hence, we can always **relabel** the blocked field σ' to be σ.

(2) In addition, we might consider that making a change of summation, or integration, variable is appropriate. In particular, we can scale the field variable. This may not be useful for a discrete spin since it then does not take values in the original set of the unblocked spin but for a field $\sigma \in \mathcal{R}$, for example, it is a possibility.

Combining these ideas and allowing for the field scaling, we write

$$\sigma'_r = Z(b)\sigma_r . \tag{71}$$

Such a rescaling does not affect the partition function $\mathcal{Z}$, but, of course, to rexpress the Hamiltonian in standard form then the u'_i will be changed in sympathy to absorb approriate powers of Z. We are always allowed to do this – it is up to us. Choosing $Z(b)$ has to do with requiring good behaviour for the u_i such as the existence of a fixed point in the RG equations. It may even be necessary in order to have a simple analysis in terms of fixed points and derive a **useful** RGE.

We then obtain the transformation

$$G(\mathbf{r},\mathbf{u}) \;=\; Z(b)^2\, G(b^{-1}\mathbf{r},\mathbf{u}') \, . \tag{72}$$

I.e.,

$$\begin{array}{llll} & \langle\sigma_{20}\sigma_0\rangle_{\mathbf{u}} \;=\; \langle\sigma'_{10}\sigma'_0\rangle_{\mathbf{u}'} & & \textbf{before} \text{ relabelling and scaling,} \\ \Longrightarrow & & & \\ & \langle\sigma_{20}\sigma_0\rangle_{\mathbf{u}} \;=\; Z(b)^2\langle\sigma_{10}\sigma_0\rangle_{\mathbf{u}'} & & \textbf{after} \text{ relabelling and scaling.} \end{array} \tag{73}$$

Here $r' = 10, r = 20$, these are the **same** physical lengths but in **different units**: $r'a' = ra$.

At a fixed point $\mathbf{u}^*$ we have $\mathbf{u}^* = \mathbf{R}(\mathbf{u}^*)$ and hence, in particular, that

$$\xi' \;=\; \xi \;=\; b^{-1}\xi \quad\Rightarrow\quad \xi \;= 0, \infty \; .$$

We are interested in the non-trivial fixed points where $\xi = \infty$. Such fixed points (f.p.) are associated with continuous phase transitions.

The RG transformation is defined by $\mathbf{u} \to \mathbf{u}' = \mathbf{R}(\mathbf{u})$. Consider the neighbourhood of a fixed point $\mathbf{u}^*$ and let

$$\mathbf{u} = \mathbf{u}^* + \mathbf{v} \; , \quad \mathbf{u}' = \mathbf{u}^* + \mathbf{v}' \; .$$

with $\mathbf{v}$ and $\mathbf{v}'$ small. Then

$$\begin{array}{ll} & u^*_i + v'_i \;=\; R_i(\mathbf{u}^* + \mathbf{v}) \;=\; R_i(\mathbf{u}^*) + \left.\dfrac{\partial R_i}{\partial u_j}\right|_{\mathbf{u}^*} v_j \\ \Rightarrow & \\ & v'_i \;=\; K_{ij}(\mathbf{u}^*)\, v_j \; , \qquad K_{ij}(\mathbf{u}^*) = \left.\dfrac{\partial R_i}{\partial u_j}\right|_{\mathbf{u}^*} \end{array} \tag{74}$$

The transformation linearizes as we saw before in the 1D Ising example. We shall assume that K_{ij} is a symmetric matrix and so is diagonalizable with real eigenvalues. Other situations might occur. Let the eigenvalues of K_{ij} be y_α with eigenvectors $\mathbf{e}_\alpha$. Then we have

$$\mathbf{v} \;=\; \sum_\alpha h_\alpha \mathbf{e}_\alpha \;, \quad \mathbf{v}' \;=\; \sum_\alpha h'_\alpha \mathbf{e}_\alpha \quad \Rightarrow \quad h'_\alpha \;=\; y_\alpha h_\alpha \;. \tag{75}$$

This last equation is really a difference equation and it is better to write it as

$$h'_\alpha \;=\; b^{\lambda_\alpha} h_\alpha \;, \qquad \lambda_\alpha = \frac{\log y_\alpha}{\log b} \;. \tag{76}$$

Each $\lambda_\alpha > 0$ determines a critical exponent. The h_α are the scaling fields (or variables) of the system and measure the distance to the f.p. along the eigenvectors $\mathbf{e}_\alpha$. We see from the 1D Ising example that when small, (t, h) are scaling fields. This is generally the case for all similar models in any dimension.

The Hamiltonian in this region can be rewritten as

$$\begin{aligned} H(\mathbf{u}) \;&=\; H(\mathbf{u}^*) \;+\; \sum_i v_i O_i(\{\sigma\}) \;=\; H(\mathbf{u}^*) \;+\; \sum_i h_\alpha O_\alpha(\{\sigma\}) \\ O_\alpha(\{\sigma\}) \;&=\; \sum_i (\mathbf{e}_\alpha)_i O_i(\{\sigma\}) \;. \end{aligned} \tag{77}$$

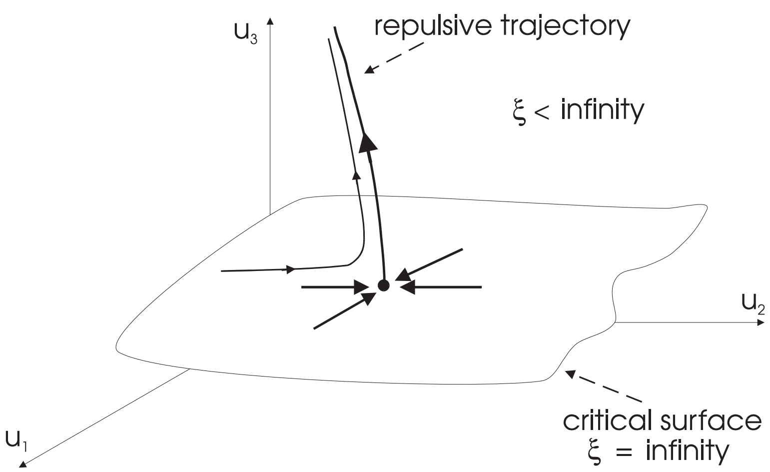
The $O_\alpha(\{\sigma\})$ are scaling operators and are special linear combinations of the operators used to define the Hamiltonian.

(i) $\lambda_\alpha > 0$. $|h_\alpha|$ increases under blocking and the system flows away from the fixed point in the $\mathbf{e}_\alpha$ direction. Such a variable is a **relevant** variable and the associated operator $O_\alpha(\{\sigma\})$ is a **relevant** operator.

(ii) $\lambda_\alpha < 0$. $|h_\alpha|$ decreases under blocking and whatever its initial value it will flow to its fixed point value. Such variables and their associated operators are **irrelevant**.

(iii) $\lambda_\alpha = 0$. Not considered.

The scenario for RG flows is



The surface of inflowing paths is called the **critical surface**. On this surface $\xi = \infty$ since all models that start in the critical surface are RG equivalent to the fixed point theory $\mathbf{u} = \mathbf{u}^*$ which has $\xi = \infty$. The corollary is that models that do not lie in the critical surface have $\xi < \infty$ since they are on trajectories that ultimately flow to a trivial fixed point which has $\xi = 0$.

The important points are

(1) Every model in the critical surface is at a continuous phase transition since $\xi = \infty$.

(2) Every such phase transition is RG equivalent to the phase transition described by the model at the fixed point where $\mathbf{u} = \mathbf{u}^*$. In particular, we will expect to see the **same** critical behaviour in all these models. This is the idea of **universality**.

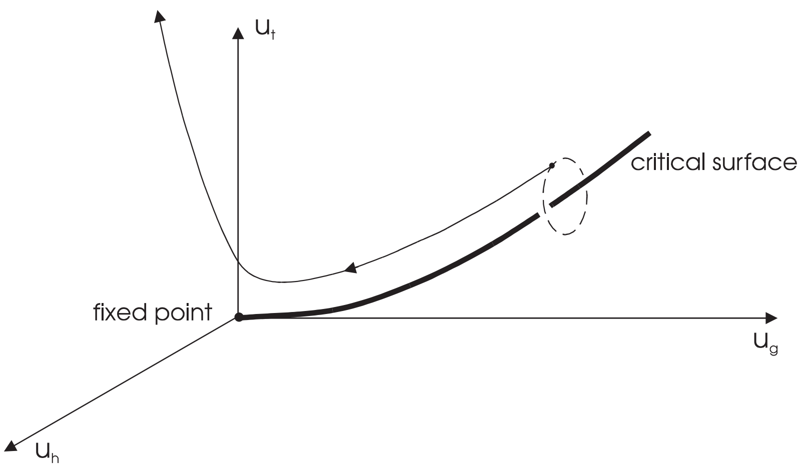
(3) To see the critical behaviour we must tune the external fields so that the couplings $\mathbf{u}$ lie in the critical surface. From before we know that the number we need to tune is κ, the codimension, of the manifold of transitions. From the RG picture we see that the number of parameters to be tuned is the number of **relevant** variables. Thus

$$\kappa \; = \; \text{number of relevant variables at the fixed point.}$$

(4) The model may contain many couplings but the majority will be irrelevant. If there are M couplings then the critical surface has dimension $M - \kappa$. In general, κ will be small for practical purposes.

Consider models in the Ising universality class and consider an ordinary critical point which has $\kappa = 2$. We have external fields (T, h) and there also may be other fields which we denote generically by g. The coupling constants are functions of (T, h, g): $\mathbf{u}(T, h, g)$.

For critical behaviour we must tune to $T = T_c(g), h = 0$. Note, that T_c is generally a function of the couplings: the value of T_c is different for different points in the critical surface – it is **not** universal. For simplicity we include just one other coupling, g, and then the space spanned by $\mathbf{u}$ has dim $= 3$. The critical surface is a line since it has dim $= 1$, and note that it lies in the surface $h = 0$. The scenario is now



We have set $\mathbf{u} = \mathbf{u}^* \; + \; (u_t, u_h, u_g)$ so that when (u_t, u_h, u_g) is small, and hence $\mathbf{u}$ lies in the neighbourhood of $\mathbf{u}^*$ (where the RG is linearizable), we have

$$u_t \; \sim \; a_t t \; , \quad u_h \; \sim \; a_h h \; , \quad u_g \; \sim \; a_g g \; ,$$

where t and h are **relevant** scaling fields (as derived earlier), and g represents all the **irrelevant** scaling fields. It is simple to define $u_h = 0$ when $h = 0 \;\; \forall \, T, g$.

The dotted circle is an example of a set of near critical theories. We follow the blocking history of one such theory.

The RG equations for blocking by a scale factor b are

$$\begin{aligned} \mathbf{u}_p \; \to \; \mathbf{u}_{p+1} \; &= \; \mathbf{R}(\mathbf{u}_p) \, , \\ F(\mathbf{u}_p, C_p) \; &= \; f(\mathbf{u}_p) \; + \; C_p \\ f(\mathbf{u}_0) \; &= \; b^{-pD} f(\mathbf{u}_p) \; + \; \sum_{j=0}^{p-1} b^{-jD} g(\mathbf{u}_j) \, . \end{aligned}$$

(1) Since $t = (T - T_c)/T_c$ and h are small we see that the trajectory passes very close to $\mathbf{u}^*$ and it passes through the neighbourhood where the RG is **linear** before it moves away from the critical surface.

(2) The flow reaches the neighbourhood of $\mathbf{u}^*$ after a **finite** number of iterations. This is because this is all it takes for the **irrelevant** variables, like g, to become close to their fixed point values. Let the number of iterations to this point be $\bar{p}$. Note that g essentially measures how far in the critical surface the model is from $\mathbf{u}^*$, and so determines the value of $\bar{p}$.

Look back at the flows for the 1D Ising model above. I did 1000 iterations per curve but all of them get to (in that case) the trivial fixed points at $x = 1$ in only a few steps indicating that $\bar{p} \sim 5 - 10$.

Then we have

$$F(\mathbf{u}_0, C_0) \; = \; b^{-\bar{p}D} F(\mathbf{u}_{\bar{p}}, C_{\bar{p}}) \; .$$

(i) $\mathbf{u}_{\bar{p}}$ is in the neighbourhood of $\mathbf{u}^*$ and so we write $\mathbf{u}_{\bar{p}} = \mathbf{u}^* + (u_t, u_h, u_g)$.

(ii) I can always choose C_0 so that $C_{\bar{p}} = 0$. Of course, this means that $C_0 = C_0(g)$ since $\bar{p}$ depends on g. However, C_0 is **insensitive** to (t, h) since these variables are very small and C_0 is certainly well behaved as $t, h \to 0$. (Another way to see this is to consider the trajectory **with** $t = h = 0$. It lies in the critical surface but the flow is always very close to the one we are studying and so C_0 is essentially the same for both.)

(iii) $b^{-\bar{p}D} \equiv K$ is constant which is not important for our purposes.

We then have

$$F(\mathbf{u}_0, C_0) \; = \; K f(u_t, u_h, u_g) \; ,$$

where $\mathbf{u}^*$ has been absorbed into a trivial redefinition of f. Because t and h are small, this blocked model is in the linear region of the RG and so we have

$$u_t \; = \; a_t t \; , \quad u_h \; = \; a_h h \; , \quad u_g \; = \; a_g g \; .$$

We can always make this choice since the basis at $\mathbf{u} = \mathbf{u}^*$ is the set of eigenvectors whose scaling fields $\{h_\alpha\}$ are (t, h, g).

The outcome is that the free energy of the original model close to the critical point can be written

$$F(\mathbf{u}_0, C_0) \;=\; Kf(a_t t, a_h h, a_g g)\;. \tag{78}$$

Perform p further iterations of the RG equations. We find

$$F(\mathbf{u}_0, C_0) \;=\; b^{-pD} f(b^{\,p\lambda_t} t, b^{\,p\lambda_h} h, b^{\,p\lambda_g} g) \;+\; \sum_{j=0}^{p-1} b^{-jD} g(b^{\,j\lambda_t} t)\;, \tag{79}$$

where we have set $K = 1, a_t = a_h = a_g = 1$ for simplicity.

Note that the inhomogeneous part is now only dependent on t and **not** h or any other coupling. This is certainly true of the 1D Ising RG equation for C_p in the linear region. RG schemes that do not have this property are not necessarily wrong but they are not useful: it can be considered a requirement of a good scheme. The physical reason why it is generally true is that external fields, such as h, couple to the large scale modes only, and so do not modify the contribution to the free energy coming purely from the small scale modes. In a good RG scheme it is only the latter over which we integrate when thinning modes and hence only these modes that determine the inhomogeneous function $g(\mathbf{u})$ which is therefore is independent of h etc. How this argument works is easier to verify for some schemes than others.

One iteration changes the scale by a discrete amount b. However, in the neighbourhood of $\mathbf{u}^*$ the point $\mathbf{u}_p$ is a slow moving as a function of p and the trajectory is essentially continuous. In fact, many RG schemes (see later) are continuous and the RG equations are differential equations and, in any case, discrete schemes can be extended to be continuous. Thus, we can treat $b^{\,p}$ and $b^{\,j}$ as continuous variables.

Denote the **total** rescaling by $\hat{b} = b^{\,p}$. Consider first the singular contribution to F (79)

$$F_s \;=\; \hat{b}^{-D} f(\hat{b}^{\,\lambda_t} t, \hat{b}^{\,\lambda_h} h, \hat{b}^{\,\lambda_g} g)\;. \tag{80}$$

Choose $\hat{b}$ so that

$$\hat{b}^{\,\lambda_t} t \;=\; 1\;,\;\; t > 0\;, \qquad \hat{b}^{\,\lambda_t} t \;=\; -1\;,\;\; t < 0\;. \tag{81}$$

That is, we iterate until we reach a reference model which, by definition, has $|t| = 1$ and relate all quantities to their values in this model. Since $\lambda_g < 0$ we have $\hat{b}^{\,\lambda_g} g \sim 0$ for $|t|$ sufficiently small. Then

$$F_s \;=\; |t|^{D/\lambda_t} f\left(\pm 1, \frac{h}{|t|^{\lambda_h/\lambda_t}}, 0\right)\;. \tag{82}$$

We see that we have recovered the scaling hypothesis postulated earlier with

$$f_{\substack{>\\<}}(x) \;=\; f(\pm 1, x, 0) \;\equiv\; f_\pm(x)\;. \tag{83}$$

We can read off the required exponents

$$\begin{aligned}
\alpha &= 2 - D/\lambda_t \ , \\
\Delta &= \frac{\lambda_h}{\lambda_t} \ , \\
\beta &= 2 - \alpha - \Delta \ = \ \frac{D - \lambda_h}{\lambda_t} \ , \\
\gamma &= 2\Delta - (2 - \alpha) \ = \ \frac{2\lambda_h - D}{\lambda_t} \ , \\
\delta &= \frac{\Delta}{\beta} \ = \ \frac{\lambda_h}{D - \lambda_h} \ .
\end{aligned} \tag{84}$$

Note that the exponents are the **same** for $t \gtrless 0$, the difference is in the coefficient only. This is obvious from the linear form of the RG equations near the fixed point.

We must check that the inhomogeneous part does not upset these predictions. In fact, it will only possibly affect α since all other indices are associated with quantities obtained by differentiating with respect to h and this term is **independent** of h as discussed above. This is a general result.

Let $s = b^{\,j\lambda_t}|t|$. The inhomogeneous part of (79) is then approximated by an integral

$$\begin{aligned}
\sum_{j=0}^{p-1} b^{-jD} g(b^{\,j\lambda_t} t) \ &\sim \ |t|^{D/\lambda_t} \int ds \left| \frac{ds}{dj} \right|^{-1} s^{-D/\lambda_t} \ g(\pm s) \ , \\
&= \ \frac{|t|^{D/\lambda_t}}{\lambda_t \log b} \int_{|t|}^{1} ds \ s^{-D/\lambda_t - 1} g(\pm s) \ .
\end{aligned} \tag{85}$$

The overall factor of $|t|^{D/\lambda_t}$ is the same as the singular contribution, but the integral has lower limit $|t|$ and we cannot be sure that it converges as $|t| \to 0$. However, it can be shown/argued that this term is finite and takes the form

$$C_\pm |t|^{D/\lambda_t} + I_\pm(t) \ , \tag{86}$$

where $I_\pm(t)$ is a power series in $|t|$ (i.e., an analytic function):

$$I_\pm(t) \equiv I_\pm^{(0)} + I_\pm^{(1)} |t| + \dots \ . \tag{87}$$

(However, the 1D Ising model is a counter-example and in fact the scaling relation $\alpha + 2\beta + \gamma = 2$ does **not** hold – see problem sheet.). This expansion effectively relies on $g(u)$ being analytic in the sense that we can expand $g(u) = \sum_n g_n u^n$ (see Cardy p56). Many books wrongly omit a proper discussion of $I_\pm(t)$ at this stage. Then

$$F_\pm(\mathbf{u}_0, C_0) \ = \ |t|^{D/\lambda_t} \left(f_\pm \left(\frac{h}{|t|^{\lambda_h/\lambda_t}} \right) + C_\pm \right) \ + \ I_\pm(t) \ . \tag{88}$$

The singular nature of observables is carried by the first term (D/λ_t an integer is a special case). The terms $C_\pm |t|^{D/\lambda_t} + I_\pm(t)$ only contribute to the specific heat exponent; $I_\pm(t)$ contributes only a non-leading non-singluar contribution and so is generally omitted from the discussion. However, when we discuss the Gaussian model we shall see that this is correct only for $D < D_c$ ($D_c = 4$ in this analysis) and that $I_\pm(t)$ actually implements the mean-field predictions for $D > D_c$.

Note that $f_\pm$ are universal functions since they are computed only from the fixed point model. Then amplitude ratios for observables are **universal**. E.g., for the ratio of susceptibilities $\chi_\pm \sim A_\pm |t|^{-\gamma}$:

$$\frac{A_+}{A_-} \;=\; \frac{f''_+(0)}{f''_-(0)} \; .$$

We need the ratio since we set $K = 1$ for simplicity above but it is **not** universal and only in the ratio does it cancel out. In a similar manner the amplitude ratio

$$\frac{f_+(0) + C_+}{f_-(0) + C_-} \; , \tag{89}$$

which contributes to the specific heat, is also universal.

When we discussed the scaling hypothesis we proposed that $[M] = D_M, [A] = -D$. By this we mean that under rescaling by a factor k the following transformations hold:

$$\begin{array}{rcrcl} a & \to & a' & = & k^{-1}\, a, \\ \mathbf{r} & \to & \mathbf{r}' & = & k\,\mathbf{r}, \\ M & \to & M' & = & k^{D_M}\, M, \\ A(M) & \to & A'(M') & = & k^{-D} A(M) \; . \end{array} \tag{90}$$

where a is the length unit in which distance are measured. Under the RG rescaling we measure distances in lattice units and we have $a \to a' = ba$ and so we identify $k = b^{-1}$ in the above. Then we can read the RG scaling transformations from the results above:

$$\begin{array}{rclcllcl} \mathbf{r} & \to & \mathbf{r}' & = & b^{-1}\mathbf{r} & \Rightarrow & [\mathbf{r}] = 1, \\ t & \to & t' & = & b^{\lambda_t} t & \Rightarrow & [t] = -\lambda_t, \\ h & \to & h' & = & b^{\lambda_h} h & \Rightarrow & [h] = -\lambda_h, \\ F(t,h) & \to & F(t',h') & = & b^D F(t,h) & \Rightarrow & [F] = -D, \\ \sigma & \to & \sigma',\ \sigma'_r & = & Z(b)\, \sigma_r, & & Z(b) \sim b^{\,\zeta} \; . \end{array} \tag{91}$$

The result for F is the usual statement for how F scales and is clearly incorporated in the scaling of F_s in (80). It also follows from the result for $F_\pm(t,h)$ in (88) since we have $[F_\pm] = [t] \cdot D/\lambda_t = -D$.

We then also have that

$$M \;\sim\; |t|^\beta \quad \Rightarrow \quad [M] = \beta\,[t] = -(D - \lambda_h) \quad \Rightarrow \; D_M = -D + \lambda_h \; .$$

Note that

$$[hM] \;=\; -\,D + \lambda_h + [h] \;=\; -\,D \; ,$$

and hence that the relevant term in the Hamiltonian for the effective field theory is

$$H_h \;=\; \int d^D r\; hM \;=\; \int d^D r'\; b^D hM \;=\; \int d^D r'\; h'M' \quad \Rightarrow \quad [H_h] = 0 \; .$$

This is correct since, as H appears as an exponent in the exponential for the partition function, it must be **dimensionless**.

An example is given by the susceptibility

$$\begin{array}{rrcl} & \chi &=& \dfrac{\partial M}{\partial h} \;\Rightarrow\; [\chi] \;=\; [M]-[h] \;=\; -D+2\lambda_h \\ \text{but} & \chi &\sim& |t|^{-\gamma} \;\Rightarrow\; [\chi] \;=\; -\gamma\,[t] \;=\; \gamma\lambda_t \\ \text{and so} & \gamma &=& (2\lambda_h - D)/\lambda_t\;, \end{array}$$

as derived earlier.

You might ask how I can simply assign a dimension to observables? The answer is

(i) it is only possible near a continuous phase transition for which the trajectories pass close to a non-trivial fixed point where we can linearize the RG equations;

(ii) ordinary dimensional analysis **is** really working but there is a hidden variable carrying dimension in the final results. E.g., in statistical physics it is the lattice spacing a and in QFT it is the large momentum cutoff Λ. This occurs in the theory of non-linear equations, fluids etc where the scale is the physical size of the source or the scale of stirring. Basically, functions simplify near a phase transition and for example the two field correlator behaves like

$$G(r) \;\sim\; \frac{a^{\eta}}{r^{D-2+\eta}}\;, \quad r \ll \xi. \tag{92}$$

G always has engineering dimension $D-2$ but the large r behaviour **looks like** it has dimension $D-2+\eta$ since the numerator balances the anomalous term.

Consider the correlation length ξ. After blocking $\bar{p}$ times to the linear region we have $\xi_{\bar{p}} = b^{-\bar{p}}\,\xi$. We then subsequently rescale by an amount $\hat{b} = |t|^{-1/\lambda_t}$ to reach the reference model with $|t| = 1$. Let the correlation length of this reference model be ξ_0. Then clearly

$$\xi \;=\; b^{\bar{p}}\,\hat{b}\,\xi_0 \;\sim\; |t|^{-1/\lambda_t}\;, \tag{93}$$

since $\bar{p}$ is small and finite and ξ_0 is fixed. But

$$\xi \sim |t|^{-\nu} \quad\Rightarrow\quad \nu \;=\; \frac{1}{\lambda_t}\;. \tag{94}$$

Putting all the results together we have the scaling relations

$$\alpha + 2\beta + \gamma = 2\;, \qquad \beta\delta = \beta + \gamma\;, \qquad \alpha = 2 - D\nu\;. \tag{95}$$

To determine η we need to discuss the spin rescaling factor $Z(b)$ in (72). The choice of Z is up to us and we must find the right criterion. In the 1D example $Z = 1$ is fine but we shall see how it is chosen in a more general example. Given Z we can deduce η as follows.

For $a \ll r \ll \xi$ we consider the model after blocking $\bar{p}$ times. We need consider only $\xi = \infty$ since $r \ll \xi$ and then $\mathbf{u}_{\bar{p}} \sim \mathbf{u}^*$. After $\bar{p}$ blockings we have

$$G(r,\mathbf{u}) \;=\; K(\bar{p})^2\,G(b^{-\bar{p}}r,\mathbf{u}^*) \;\;\text{and}\;\; K(\bar{p}) = Z(b,\mathbf{u}_1)Z(b,\mathbf{u}_2)\ldots Z(b,\mathbf{u}_{\bar{p}}). \tag{96}$$

Here $K(\bar{p})$ accumulates the Z factors on the flow into the fixed point and $\mathbf{u}_{\bar{p}} \sim \mathbf{u}^*$ by definition. Now define

$$\hat{G}(r) \ \equiv \ G(r, \mathbf{u}^*) \quad \text{and} \quad \bar{G}(r) \ = \ K(\bar{p})^2 \hat{G}(b^{-\bar{p}} r) \ , \tag{97}$$

where $\hat{G}(r)$ is a **universal** function since it is defined for $\mathbf{u} = \mathbf{u}^*$. Let $Z(b, \mathbf{u}^*) = b^\zeta$ and then, remembering that $\bar{p}$ is fixed and finite, after further blocking p times with $\hat{b} = b^p$ we have

$$G(r, \mathbf{u}) \ = \ \bar{G}(r) \ \underbrace{=}_{p \text{ blockings}} \ Z^{2p}\, \bar{G}(b^{-p} r) \ = \ \hat{b}^{2\zeta}\, \bar{G}(\hat{b}^{-1} r) \ \Rightarrow \ \bar{G}(r) \ \sim \ \frac{1}{r^{-2\zeta}} \ . \tag{98}$$

Thus

$$D - 2 + \eta \ = \ -2\zeta \quad \Rightarrow \quad \eta \ = \ 2 - D - 2\zeta \ . \tag{99}$$

For the $1D$ Ising model $\quad \zeta = 0 \ , \quad \Rightarrow \quad \eta = 1$.

Now, for a function $f(r)$ that has $[f] = \rho$ we have that $f(r) = b^{\,\rho} f(b^{-1} r)$. An important deduction is that we can assign a dimension to the Green function from above: $[G] = 2\zeta$. In turn this implies a dimension for the field σ_r: $[\sigma] = \zeta$ since $G(r) = \langle \sigma_r \ \sigma_0 \rangle$. In other words, all correlation functions have a dimension determined by the field dimension: $[G^{(n)}] = n\zeta$. This analysis applies whenever $r \ll \xi$ and is meaningful because $\xi \to \infty$ as we approach a critical surface.

12 The Partition Function and Field Theory

The partition function $\mathcal{Z}$ is defined by

$$\mathcal{Z} = \int \{d\phi\} e^{-H(\phi)} \tag{100}$$

where $\phi(\mathbf{x})$ is a generic field degree of freedom. $H(\phi)$ is the effective Hamiltonian given by

$$H(\phi) = \int_{\Lambda^{-1}} d\mathbf{x} \ \mathcal{H}(\phi(\mathbf{x}))$$

where $\mathcal{H}(\phi(\mathbf{x}))$ is the Hamiltonian density. Λ is the large momentum cut-off which, for a lattice of spacing a, is $\Lambda = 2\pi/a$. In this case the integral will be a sum over all sites of a discrete Hamiltonian density. The **crucial** point is that there will, in general, be a cut-off of some kind. The integral stands for the sum over sites in the lattice case:

$$\int dx \leftrightarrow \sum_n \ , \qquad \int \frac{dk}{2\pi} \leftrightarrow \sum_k \ . \tag{101}$$

For a single scalar field $\mathcal{H}$ takes the general form

$$\mathcal{H}(\Lambda, \phi) = \frac{1}{2}\alpha^{-1}(\Lambda, T)(\nabla\phi)^2 + \frac{1}{2}m^2(\Lambda, T)\phi^2 + \frac{1}{4!}g(\Lambda, T)\phi^4 \ + \ldots + (\nabla\phi)^4 + \ldots \ .$$

The dependence on T is encoded now through the coupling constants. We shall only discuss the ordinary critical point at $h = 0$ otherwise we need a generalization to include e.g., ϕ and ϕ^6 terms. The coupling constants would then be functions of all the external fields that can be tuned to realize the various phase transitions of the theory.

The kinetic term no longer has unit coefficient since it is not a more special term than any other. However, it is often convenient to rescale ϕ to make this particular coefficient unity.

The coupling constants depend on the cutoff Λ (or equivalently, a) since when we block the theory the couplings **change** according to the RG equations as does Λ: $\Lambda \to \Lambda' = \Lambda/b$. The dependence on Λ parameterizes the RG trajectory.

This general structure can be justified by a number of arguments:

(1) $\mathcal{H}$ is the result of iterating *RG* equations for an original model which is in the **same** universality class as the scalar field theory. The large-scale degree of freedom which characterizes the critical behaviour is a scalar field e.g., the magnetization in a ferromagnet or the density in a liquid/gas system.

For example, the blocking transformation we can choose

$$\hat{\phi}(\mathbf{x}) = \frac{1}{L^D} \int_v d\mathbf{x}'\ \phi(\mathbf{x}'),$$

where v is a volume centred at $\mathbf{x}$, L is the block size and $v = L^D > a^D$.

Alternatively, we may choose

$$\hat{\phi}(\mathbf{x}) = \int_{0 \le p \le \Lambda'} \frac{d^D p}{(2\pi)^D}\ e^{i\mathbf{p}\cdot\mathbf{x}}\ \tilde{\phi}(\mathbf{p}).$$

Here $\hat{\phi}(\mathbf{x})$ is composed of the low-momentum degrees of freedom only : $L = 2\pi/\Lambda'$.

In either case the blocked model is equivalent to a field theory for $\hat{\phi}$ defined on a lattice of spacing $\sim L$. Then

$$e^{-H(\Lambda',\hat{\phi})} \;=\; \int d\phi\ \delta\left(\hat{\phi} - \hat{\phi}(\phi)\right)\ e^{-H(\Lambda,\phi)}$$

(2) We saw that mean field theory gives an expression for the free energy in terms of a scalar field e.g., M. The mean field approximation consisted of ignoring fluctuations as irrelevant. However, we should expect M to vary on the scale of the correlation length and so a more general expression for the energy will include kinetic terms like $(\nabla\phi)^2$ as shown above. These terms like $(\nabla\phi)^4$ are generally irrelevant (in RG terms) and will be ignored.

(3) We probe a system with a long wavelength external field $J(\mathbf{x})$. The response to changes in J is the variation in the conjugate scalar field, denoted ϕ here. This field will vary only on the large scale and the energy can always be expanded as above for $\mathcal{H}$.

We can recover the Landau-Ginsburg theory from this formalism. As we block, the cut-off Λ decreases until $\Lambda \le \xi^{-1}$ or equivalently $a \ge \xi$. The shortest scale in the model is then the block size $L \sim \Lambda^{-1}$. On dimensional grounds LG theory assumes that for L large enough

$$\alpha^{-1}(\Lambda)(\nabla\phi)^2 \sim \Lambda^2\alpha^{-1}(\Lambda)\phi^2 \sim \frac{\alpha^{-1}(\Lambda)}{L^2}\phi^2 \ll m^2\phi^2 \tag{102}$$

and hence the **effective** Hamiltonian is insensitive to $\nabla\phi$ and so ϕ can be treated as a constant. Then

$$\begin{aligned} e^{-VF} &= \int d\phi\; e^{-VH(\Lambda=0,\phi)}, \\ H &= \frac{1}{2}m^2(0,T)\phi^2 \;+\; \frac{1}{4!}g(0,T)\phi^4 \;+\; \ldots \end{aligned}$$

We have integrated out over all scales and produced **renormalized** coupling constants and an **effective** Hamiltonian depending on ϕ.

Now

$$\begin{aligned} \int dx\; e^{-VS(x)} &= e^{-VS(x_0)} \int du\; e^{-V[\frac{1}{2}S''(x_0)u^2 \;+\; \ldots]} \\ &= e^{-VS(x_0)} \int du\; e^{-\frac{1}{2}VS''(x_0)u^2}\; [1 - \frac{1}{4!}VS^{(4)}u^4 \;+\; \ldots], \\ &= e^{-VS(x_0)} \left(\frac{2\pi}{VS''}\right)^{\frac{1}{2}} \left[1 - \frac{1}{8V}\frac{S^{(4)}}{(S'')^2} \;+\; \ldots\right], \\ \text{where}\quad S'(x_0) &= 0\;, \quad u = x - x_0\;. \end{aligned}$$

Hence

$$F/kT \;=\; H(\Lambda\!=\!0,\phi_0) \;+\; O\left(\frac{\log V}{V}\right),$$

where ϕ_0 is the **global minimum** of $H(0,\phi)$. This is **Landau's** method with $H(0,\phi)$ as the free energy function **BUT** this procedure requires the limit $\Lambda \to 0$, $(L \to \infty)$ to be under control. In particular Landau assumes that $m^2(0,T)$ is **analytic** in T. Then all that the integration over scales has done is to fix the value of T_c. This assumption is **wrong** for $D \le D_c$ and Landau's method fails.

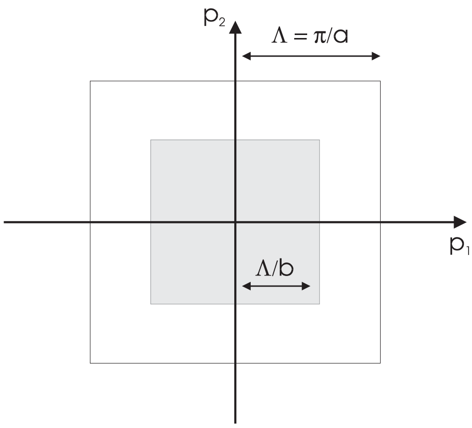
13 The Gaussian Model

A soluble field theory is the Gaussian model where H is quadratic in the fields. We have

$$\mathcal{H} \;=\; \frac{1}{2}\Big(\alpha^{-1}(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})\Big) \;-\; J(\mathbf{x})\phi(\mathbf{x})\;. \tag{103}$$

Again note that the kinetic term has a coefficient or coupling constant α^{-1}. It is better to rewrite this in wavevector space (or momentum space) using

$$\tilde{\phi}(\mathbf{p}) \;=\; \int d\mathbf{x}\; e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{x})\;. \tag{104}$$



Note that since $\phi(\mathbf{x})$ is real $\tilde{\phi}(-\mathbf{p}) = \tilde{\phi}^*(\mathbf{p})$. For an infinite volume lattice of spacing a the values of $\mathbf{p}$ lie in the Brillouin zone which in $2D$ is shown in the figure.

Then

$$H(\tilde{\phi}) = \frac{1}{2}\int_{p\leq\Lambda} \frac{d^Dp}{(2\pi)^D} \left(\alpha^{-1}p^2 + m^2\right)|\tilde{\phi}(\mathbf{p})|^2 - \left(\tilde{J}(-\mathbf{p})\tilde{\phi}(\mathbf{p}) + \tilde{J}(\mathbf{p})\tilde{\phi}(-\mathbf{p})\right), \quad p = |\mathbf{p}|. \tag{105}$$

I have arranged the $\tilde{J}\tilde{\phi}$ term so that it is explicitly real. We then have

$$\mathcal{Z}_0 = \int \prod_{\mathbf{p}} d\tilde{\phi}(\mathbf{p})\, e^{-H(\tilde{\phi})} . \tag{106}$$

The exponent is **diagonal** and we can complete the square for each $\mathbf{p}$ separately

$$\begin{aligned}
&\frac{1}{2}\tilde{\phi}(\mathbf{p})\tilde{\Delta}(\mathbf{p})\tilde{\phi}(-\mathbf{p}) - \frac{1}{2}\left(\tilde{J}(-\mathbf{p})\tilde{\phi}(\mathbf{p}) + \tilde{J}(\mathbf{p})\tilde{\phi}(-\mathbf{p})\right) = \\
&\overbrace{\frac{1}{2}\left(\tilde{\phi}(\mathbf{p}) - \tilde{J}(\mathbf{p})\tilde{\Delta}^{-1}(\mathbf{p})\right)\tilde{\Delta}(\mathbf{p})\left(\tilde{\phi}(-\mathbf{p}) - \tilde{J}(-\mathbf{p})\tilde{\Delta}^{-1}(\mathbf{p})\right)}^{\mathcal{H}_G(\tilde{\phi})} \\
&\qquad - \frac{1}{2}\tilde{J}(\mathbf{p})\tilde{\Delta}^{-1}(\mathbf{p})\tilde{J}(-\mathbf{p}) , \\
&\tilde{\Delta}(\mathbf{p}) = \alpha^{-1}p^2 + m^2 .
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{Z}_0 &= \mathcal{Z}_G \exp\left(\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\, \tilde{J}(\mathbf{p})\tilde{\Delta}^{-1}(\mathbf{p})\tilde{J}(-\mathbf{p})\right) , \qquad (107) \\
\mathcal{Z}_G &= \int \{d\tilde{\phi}\} \exp\left(\int \frac{d^Dp}{(2\pi)^D}\, \mathcal{H}_G(\tilde{\phi})\right) .
\end{aligned}$$

$\mathcal{Z}_G$ is **independent** of $\tilde{J}$ since it is a purely Gaussian integral which can be done by a change of variables in the usual way. Hence, $\mathcal{Z}_G$ is an overall constant multiplier which can usually be ignored. We shall do this in all that follows.

As seen before we have

$$\langle\phi(\mathbf{x})\rangle = \left.\frac{\delta}{\delta J(\mathbf{x})}\right|_{J=0} \log \mathcal{Z}_0 , \qquad \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle_c = \left.\frac{\delta^2}{\delta J(\mathbf{x})\,\delta J(\mathbf{y})}\right|_{J=0} \log \mathcal{Z}_0 . \tag{108}$$

We also have

$$\langle\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{p})\rangle_c \;=\; \int d\mathbf{x}\, d\mathbf{y}\; e^{-i\mathbf{q}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}}\, \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle_c \;.$$

Using translation invariance $\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle_c = G_0(\mathbf{r})$, $\mathbf{r} = \mathbf{y} - \mathbf{x}$, and so, changing variables to $\mathbf{x}$ and $\mathbf{r}$,

$$\langle\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{p})\rangle_c = \int d\mathbf{x}\, d\mathbf{r}\; e^{-i(\mathbf{q}+\mathbf{p})\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{r}} G_0(\mathbf{r}) = (2\pi)^D \delta^{(D)}(\mathbf{p}+\mathbf{q})\, \tilde{G}_0(\mathbf{p}) \;.$$

Thus

$$\langle\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{p})\rangle_c \;=\; (2\pi)^D \delta^{(D)}(\mathbf{p}+\mathbf{q})\, \tilde{G}_0(\mathbf{p}) \;, \tag{109}$$

but from (105) and (107) above

$$\begin{aligned}
\langle\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{p})\rangle_c \;&=\; (2\pi)^{2D} \left.\frac{\delta^2}{\delta\tilde{J}(-\mathbf{q})\,\delta\tilde{J}(-\mathbf{p})}\right|_{\tilde{J}=0} \log \mathcal{Z}_0 \;=\; (2\pi)^D \delta^{(D)}(\mathbf{p}+\mathbf{q})\, \tilde{\Delta}(\mathbf{p})^{-1} \\
\Rightarrow \quad \tilde{G}_0(\mathbf{p}) \;&=\; \tilde{\Delta}^{-1}(\mathbf{p}) \;=\; \frac{\alpha}{p^2 + \alpha m^2} \;.
\end{aligned} \tag{110}$$

[It helps to note that

$$\tilde{J}(\mathbf{p}) = \int d^D q\; \delta(\mathbf{p}+\mathbf{q})\; \tilde{J}(-\mathbf{q}) \;, \qquad \tilde{\Delta}(\mathbf{p}) = \tilde{\Delta}(-\mathbf{p}) \;. \qquad]$$

We immediately have

$$G_0(r) \;=\; \int \frac{d^D p}{(2\pi)^D}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{\alpha}{p^2 + \alpha m^2} \;\sim\; \begin{cases} \dfrac{\xi e^{-r/\xi}}{(r\xi)^{(D-1)/2}} & a \ll \xi \ll r \;, \\[2ex] \dfrac{1}{r^{(D-2)}} & a \ll r \ll \xi \;, \end{cases}$$

with $\xi^{-2} = \alpha m^2$. Thus for $\alpha = 1$ we have $\xi = 1/m$.

Also, from earlier, the susceptibility is given by the general result

$$\begin{aligned}
\chi \;&=\; \int d\mathbf{x}\, G(r) \;=\; \tilde{G}(0) \quad \Rightarrow \\
\chi \;&=\; \frac{1}{m^2} \;=\; \alpha\xi^2 \;.
\end{aligned} \tag{111}$$

In an interacting theory we can block until $\Lambda\,\xi = \kappa \ll 1$ and then the interactions between the block spins are heavily suppressed and the theory is well described by a Gaussian model with effective coupling constants $\alpha(\Lambda = \kappa\,\xi^{-1}, T), m(\Lambda = \kappa\,\xi^{-1}, T)$. From above we see that as ξ diverges at a critical point so does χ. We might expect from this that since $\xi \sim |t|^{-\nu}$ and then $\chi \sim |t|^{-2\nu}$. However, we must allow that $\alpha(\kappa\,\xi^{-1}, T) \sim \xi^\rho \;\Rightarrow\; \chi \sim |t|^{-(2+\rho)\nu}$ instead.

To investigate such behaviour further we implement a RG transformation which consists of two steps: a thinning of high momentum degrees of freedom and a rescaling step, just as we did in earlier cases. We illustrate this procedure by applying to the Gaussian model.

We set $J(\mathbf{x}) \equiv h$, a constant. Then the relevant term in $\mathcal{H}$ is

$$h \int d\mathbf{x}\; \phi(\mathbf{x}) \;=\; h\tilde{\phi}(0) \;. \tag{112}$$

1) **Thinning.** The blocking strategy is to divide $\tilde{\phi}$ into two parts:

$$
\begin{aligned}
\tilde{\phi}(\mathbf{p}) &= \tilde{\phi}_<(\mathbf{p}) + \tilde{\phi}_>(\mathbf{p}) \\
\tilde{\phi}_<(\mathbf{p}) &= \begin{cases} \tilde{\phi}(\mathbf{p}) & 0 \le p \le \Lambda/b \\ 0 & \Lambda/b < p \le \Lambda \end{cases} \\
\tilde{\phi}_>(\mathbf{p}) &= \begin{cases} 0 & 0 \le p \le \Lambda/b \\ \tilde{\phi}(\mathbf{p}) & \Lambda/b < p \le \Lambda \end{cases}
\end{aligned}
$$

This divides the Brillouin zone into two regions associated with low momentum and high momentum, respectively, using a scale factor b. We can then decompose the partition function into a piece that depends on $\tilde{\phi}_<$ only times a piece that depends on $\tilde{\phi}_>$ only. This is easily done because the field theory is gaussian. We get

$$
\mathcal{Z}_0 \;=\; \int \prod_{\mathbf{p}} d\tilde{\phi}_< \, e^{-H(\tilde{\phi}_<)} \int \prod_{\mathbf{p}} d\tilde{\phi}_> \, e^{-H(\tilde{\phi}_>)} \tag{113}
$$

The integrals over $\tilde{\phi}_>$ can be explicitly done since they are gaussian and they simply give an overall factor in the partition function which encodes their contribution to the free energy. Only $H(\tilde{\phi}_<)$ depends on h since h only couples to $\tilde{\phi}(0) \equiv \tilde{\phi}_<(0)$.

$$
\mathcal{Z}_0 \;=\; e^{-\beta F_>} \int \prod_{\mathbf{p}} d\tilde{\phi}_< \, \exp\left(-H(\tilde{\phi}_<)\right) \;. \tag{114}
$$

The theory is described by $H(\tilde{\phi}_<)$ but with a smaller cutoff $\Lambda' = \Lambda/b$. In other words we have integrated out high momentum degrees of freedom and have a new or blocked theory defined on a Brillouin zone of extent given by Λ'.

The Hamiltonian of the blocked model is, in this case, the **same** as the original one but with cut-off Λ'

$$
H(\tilde{\phi}_<) \;=\; \int_{p<\Lambda'} \frac{d^D p}{(2\pi)^D} \, \frac{1}{2} \left(\alpha^{-1} p^2 + m^2\right) |\tilde{\phi}_<(\mathbf{p})|^2 + h\tilde{\phi}_<(0) \;, \qquad p = |\mathbf{p}|, \tag{115}
$$

(2) **Rescaling**. We rescale the momentum by a factor of b so that the Brillouin zone is restored to its original size and hence the related cutoff to its original value. This is simply a change of units; in the blocking for the Ising model we always naturally used the lattice spacing as our unit and this is just doing the same. However, We can also **choose** to rescale the field variable at the same time. We thus define the rescalings

$$
\begin{aligned}
\Lambda' \;&\rightarrow\; \Lambda \;=\; b\,\Lambda' \;, \\
\mathbf{p} \;&\rightarrow\; \mathbf{q} \;=\; b\,\mathbf{p} \quad \Rightarrow \quad 0 \le q \le \Lambda \;, \\
\tilde{\phi}_<(\mathbf{p}) \;&\rightarrow\; \tilde{\phi}(\mathbf{q}) : \;\; \tilde{\phi}_<(\mathbf{p}) \;=\; \tilde{Z}\,\tilde{\phi}(\mathbf{q}) \;.
\end{aligned} \tag{116}
$$

This is just a change of field integration variable. Note that the rescaling factor, Z, for the coordinate space field $\phi(\mathbf{x})$ is given by $Z = b^{-D}\tilde{Z}$.

In terms of the new or blocked fields the blocked Hamiltonian is now

$$
H \;=\; \frac{1}{2}\tilde{Z}^2 b^{-D} \int_{q\le\Lambda} \frac{d^D q}{(2\pi)^D} \left(\alpha^{-1}\frac{q^2}{b^2} + m^2\right) |\tilde{\phi}(\mathbf{q})|^2 + \tilde{Z}h\tilde{\phi}(0) \;. \tag{117}
$$

We can relabel **q** as **p** since it is a dummy integration variable. Then we see that we recover a gaussian Hamiltonian of the same form as the original.

The important point is that the by thinning and then rescaling we have the **same** cutoff for the original and the blocked models. Of course, all quantities with dimension will scale with b as well. For example, the correlation length for the blocked Hamiltonian is smaller by a factor of b: $\xi \to \xi/b$. The RG transformation denoted by $R(b)$ is then

$$\begin{aligned} R(b) : H(\phi, \alpha, m, h, \Lambda) &\to H(\phi, \alpha', m', h', \Lambda) \ , \\ (\alpha')^{-1} &= \tilde{Z}^2 \, b^{-D-2} \alpha^{-1} \ , \\ m' &= \tilde{Z} \, b^{-D/2} m \ , \\ h' &= \tilde{Z} h \ . \end{aligned} \tag{118}$$

This defines the RG equations for α^{-1}, m, h. The question is what is most useful choice for the field rescaling factor $\tilde{Z}$? It should be emphasized that **any** choice for $\tilde{Z}$ will yield correct results but there is a particular best choice. Some examples:

1) $\tilde{Z}^2 < b^D$. This choice implies that both α^{-1}, m are **irrelevant** since they will flow to a fixed point at $\alpha^{-1} = m = 0$. This is not very useful and must extract the physics from the **way** these quantities decay to zero;

2) $\tilde{Z}^2 > b^{D+2}$. Both α^{-1}, m are **relevant** near the f.p. at $\alpha^{-1} = m = 0$. This is also tricky to analyze although it **is** a choice that might be entertained for more complicated models.

In fact the best choice is $\tilde{Z}^2 = b^{D+2}$ for which we find the RG equations

$$(\alpha')^{-1} = \alpha^{-1} \ , \quad m' = bm \ , \quad h' = b^{D/2+1} h \tag{119}$$

α is now constant and can be chosen as $\alpha = 1$. Near to a critical point $\xi = 1/m$ diverges and we can expand m^2 about $T = T_c$ to give $m^2 \sim |t|$. We thus see from (119) that

$$\begin{array}{lllllll} |t| & \to & |t'| & = & b^2 |t| & \Rightarrow & \lambda_t = 2 \ , \\ h & \to & h' & = & b^{D/2+1} h & \Rightarrow & \lambda_h = D/2 + 1 \ . \end{array} \tag{120}$$

This predicts

$$\alpha = (4-D)/2 \ , \quad \beta = (D-2)/4 \ , \quad \nu = 1/2 \ , \quad \gamma = 1 \ . \tag{121}$$

These are **not** mean field predictions but note that they **do** attain their mean field values for $D = 4$. This is an indication that $D_c = 4$. At $h = 0$ from (88)

$$F_\pm(\mathbf{u}_0, C_0) = |t|^{D/\lambda_t} \left(f_\pm(0) + C_\pm \right) + I_\pm(t) \ . \tag{122}$$

For $D > 4$ have $D/\lambda_t = D/2 > 2$ and the first term is **less** singular than the t^2 term in $I_\pm(t)$ which therefore controls the specific heat exponent giving $\alpha = 0$ which is the mean field result. The way the critical behaviour becomes mean-field behaviour for the other exponents is concerned with the idea of "dangerous irrelevant variables" but will not discuss this here.

Note that this choice for $\tilde{Z}$ corresponds to $Z^2 = b^{-D+2}$, where Z is the corresponding field rescaling (or renormalization) factor for $\phi(\mathbf{x})$.

14 The Perturbation Expansion

We start with the definition

$$< \phi(0)\phi(\mathbf{z}) >= \frac{1}{\mathcal{Z}} \int d\phi \; \phi(0)\phi(\mathbf{z}) \; e^{-S(\phi)} \; .$$

Let

$$S(\phi) \;=\; \int d\mathbf{x} \; \frac{1}{2}(\nabla\phi)^2 \;+\; \frac{1}{2}m^2\phi^2 \;+\; \frac{1}{4!}g\phi^4 \quad ,$$

where m^2 can be negative. The situation $m^2 < 0$ corresponds to spontaneous breaking of the symmetry $\phi \to -\phi$ in the bare Hamiltonian. However, we shall see that the **effective** mass is positive once loop corrections are included.

The quantities we need are the **connected** r-point functions

$$\langle \phi(\mathbf{z}_1)\phi(\mathbf{z}_2)\ldots\phi(\mathbf{z}_r)\rangle_c \;=\; \frac{\delta}{\delta J(\mathbf{z}_1)}\frac{\delta}{\delta J(\mathbf{z}_2)}\cdots\frac{\delta}{\delta J(\mathbf{z}_r)}\bigg|_{J=0} \log \mathcal{Z}(J) \; ,$$

where

$$\mathcal{Z}(J) \;=\; \int d\phi \; e^{-S(\phi)+J\phi} \; , \qquad J\phi \;\equiv\; \int d\mathbf{x} \; J(\mathbf{x})\phi(\mathbf{x}) \; .$$

We can calculate the expression for $\langle\rangle_c$ by example as we did before:

$$< \phi(\mathbf{y})\phi(\mathbf{z}) >_c \;=\; < \phi(\mathbf{y})\phi(\mathbf{z}) > -\langle\phi(\mathbf{y})\rangle\langle\phi(\mathbf{z})\rangle \; .$$

They are the physical response functions encoding the response to an external field. On physical grounds we would the expect that

$$\langle \phi(\mathbf{z}_1)\phi(\mathbf{z}_2)\ldots\phi(\mathbf{z}_r)\rangle_c \;\to\; 0 \quad |\mathbf{z}_i - \mathbf{z}_j| \to \infty \qquad \text{like} \quad e^{-|\mathbf{z}_i - \mathbf{z}_j|/\xi} \; .$$

This will not be true unless we subtract off all disconnected contributions. Note also that then

$$\int d\mathbf{z}_1 \ldots d\mathbf{z}_r \; \langle \phi(\mathbf{z}_1)\phi(\mathbf{z}_2)\ldots\phi(\mathbf{z}_r)\rangle_c \;\propto\; V \; , \tag{123}$$

as is expected on physical grounds. For the moment we consider $m^2 > 0$ and we have

$$< \phi(\mathbf{y})\phi(\mathbf{z}) >_c \;=\; \left(\frac{1}{\mathcal{Z}}\frac{\delta^2}{\delta J(\mathbf{y})\delta J(\mathbf{z})}\mathcal{Z}(J)\right) \;-\; \left(\frac{1}{\mathcal{Z}}\frac{\delta}{\delta J(\mathbf{y})}\mathcal{Z}(J)\right)\left(\frac{1}{\mathcal{Z}}\frac{\delta}{\delta J(\mathbf{z})}\mathcal{Z}(J)\right)\bigg|_{J=0} \; .$$

However, the second term on RHS just subtracts off the disconnected piece, so if we restrict our calculation of the first term to connected contributions we can omit these disconnected terms. We have

$$S(\phi) \;=\; \underbrace{\int d\mathbf{x} \; \frac{1}{2}(\nabla\phi)^2 \;+\; \frac{1}{2}m^2\phi^2}_{S_0} \;+\; \int d\mathbf{x} \; \frac{1}{4!}g\phi^4 \; ,$$

and so we consider

$$\begin{aligned} \mathcal{Z}(J) \;&=\; \underbrace{\int d\phi \; e^{-S_0(\phi)+J\phi}}_{Z_0(J)} \; e^{-\int d\mathbf{x}\frac{1}{4!}g\phi^4} \\ &=\; e^{-\frac{g}{4!}\int d\mathbf{x}\left(\frac{\delta}{\delta J(\mathbf{x})}\right)^4} \; \mathcal{Z}_0(J) \; . \end{aligned}$$

Here

$$\mathcal{Z}_0(J) \;=\; e^{\frac{1}{2}\int d\mathbf{x}\, d\mathbf{y}\; J(\mathbf{x})G(\mathbf{x}-\mathbf{y})J(\mathbf{y})} \;.$$

Then (taking only connected contributions)

$$< \phi(\mathbf{y})\phi(\mathbf{z}) >_c \;=\; \left.\frac{\frac{\delta^2}{\delta J(\mathbf{y})\delta J(\mathbf{z})} e^{-\frac{g}{4!}\int d\mathbf{x}\,\left(\frac{\delta}{\delta J(\mathbf{x})}\right)^4} \mathcal{Z}_0(J)}{e^{-\frac{g}{4!}\int d\mathbf{x}\,\left(\frac{\delta}{\delta J(\mathbf{x})}\right)^4} \mathcal{Z}_0(J)}\right|_{J=0} \;.$$

We tacitly cancel overall constant multipliers such as $\mathcal{Z}_G$ discussed before. We now expand both numerator and denominator as a series in g. We note that the denominator does not depend on $\mathbf{y}, \mathbf{z}$ and so will only contribute **disconnected** terms when the whole expression is written as a power series in g. Because they are disconnected these contribute terms diverge with the volume V and must ultimately cancel otherwise (123) would not hold . In fact, this is ensured because the denominator is a factor of the numerator. Hence, we recover the terms we want by considering **only** the connected terms from the numerator. This will be understood from now on.

The $O(g^0)$ term is just what we calculated in the Gaussian model and so we find

$$< \phi(\mathbf{y})\phi(\mathbf{z}) >_c \;=\; \left.\frac{\delta^2}{\delta J(\mathbf{y})\delta J(\mathbf{z})}\; \mathcal{Z}_0(J)\right|_{J=0} \;=\; G_0(\mathbf{z}-\mathbf{y}) \;+\; O(g) \;.$$

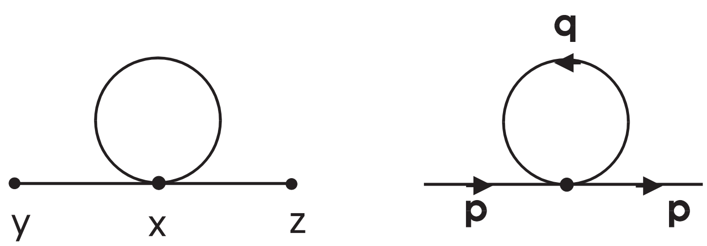
The $O(g)$ term is

$$-\frac{g}{4!}\,\frac{\delta^2}{\delta J(\mathbf{y})\delta J(\mathbf{z})}\int d\mathbf{x}\,\left.\left(\frac{\delta}{\delta J(\mathbf{x})}\right)^4 \mathcal{Z}_0(J)\right|_{J=0} \;.$$

The rules for differentiating $\mathcal{Z}_0(J)$ are clear. Every pair of derivatives gives a Gaussian two-point function, or propagator, $G_0(\mathbf{r})$. Thus we can immediately write down this contribution by adding up the results of considering all possible pairings but remembering to keep only connected contributions. This sum over all pairings is none other than **Wick's theorem**. Wick's theorem is the general way to work out Gaussian integrals such as we have here. We then find that this term is

$$=\; 4\cdot 3\left(-\frac{g}{4!}\right)\int d\mathbf{x}\; G_0(\mathbf{y}-\mathbf{x})G_0(0)G_0(\mathbf{x}-\mathbf{z}) \;.$$

This can be represented graphically as (no disconnected terms appear)



This contribution is best expressed in momentum space since it clearly consists of convolutions of functions. We then get

$$\langle\tilde{\phi}(\mathbf{p})\tilde{\phi}(\mathbf{p}')\rangle_c \;=\; (2\pi)^D\delta^{(D)}(\mathbf{p}+\mathbf{p}')\,\tilde{G}(\mathbf{p})$$

$$\tilde{G}(\mathbf{p}) \;=\; \tilde{G}_0(\mathbf{p}) \;-\; \frac{g}{2}\,\tilde{G}_0(\mathbf{p})\left(\int \frac{d^Dq}{(2\pi)^D}\,\tilde{G}_0(\mathbf{q})\right)\tilde{G}_0(\mathbf{p})$$

The $O(g)$ term is represented graphically above. We have that

$$\tilde{G}_0(\mathbf{p}) \;=\; \frac{1}{p^2+m^2}\;,$$

and so the integrals over momentum can be written explicitly and then evaluated. In the above term we have

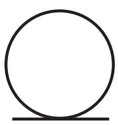
$$\int \frac{d^Dq}{(2\pi)^D}\,\tilde{G}_0(\mathbf{q}) \;=\; \int \frac{d^Dq}{(2\pi)^D}\,\frac{1}{q^2+m^2}\;.$$

We note that this is **divergent** and so the cut-off will be rather necessary!

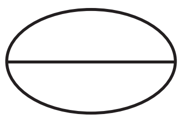
All perturbation theory terms can be calculated and represented by diagrams in this way. The Feynman rules for constructing the diagrams are

$\frac{1}{\mathbf{p}^2+m^2}$	for each propagator line
$\int \frac{d^Dq}{(2\pi)^D}$	for each closed loop
$(2\pi)^4\delta^D(\sum \mathbf{p}_i)$	momemtum conservation at each vertex
$C = \frac{1}{S}$	symmetry factor

Where S is the number of symmetry operations that leave the graph unchanged. For example, if we include also a ϕ^3 interaction we have

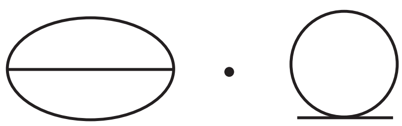


S = 2



S = 6

Note again that **disconnected** contributions to the numerator, e.g.,



are cancelled by terms in the denominator. Hence, only get **connected** graphs for expansion of G. These come from the **numerator** only.

To account for the situation where $m^2 < 0$, which is the situation we will encounter, we notice that we can redefine S_0

$$S(\phi) \;=\; \underbrace{\int d\mathbf{x}\, \frac{1}{2}(\nabla\phi)^2 \;+\; \frac{1}{2}\mu^2\phi^2}_{S_0} \;+\; \int d\mathbf{x}\, \frac{1}{2}(m^2-\mu^2)\phi^2 \;+\; \frac{1}{4!}g\phi^4 \;,$$

with $\mu^2 > 0$. This corresponds to expanding about a different unperturbed or Gaussian action. After all, in quantum mechanics the complete set of oscillator states spans the space of wavefunctions **whatever** choice is made for the oscillator frequency ω. Of course, there may be an optimal choice which could, for example, be chosen using the variational principle.

Then we have

$$\begin{aligned}
\tilde{G}(\mathbf{p}) \;&=\; \tilde{G}_0(\mathbf{p}) \;-\; \tilde{G}_0(\mathbf{p})(m^2-\mu^2)\tilde{G}_0(\mathbf{p}) \;-\; \frac{g}{2}\,\tilde{G}_0(\mathbf{p})\left(\int \frac{d^Dq}{(2\pi)^D}\,\tilde{G}_0(\mathbf{q})\right)\tilde{G}_0(\mathbf{p}) \;+\; \ldots \\
\tilde{G}_0(\mathbf{p}) \;&=\; \frac{1}{p^2+\mu^2}\;.
\end{aligned}$$

It is useful to define the **truncated** r-point function. The r-point function $\tilde{G}_r(\mathbf{p}_1,\ldots,\mathbf{p}_r)$ is defined by

$$(2\pi)^D\delta^{(D)}\left(\sum_{i=1}^{r}\mathbf{p}_i\right)\tilde{G}_r(\mathbf{p}_1,\ldots,\mathbf{p}_r) = \int d\mathbf{x}_1\ldots d\mathbf{x}_r \;<\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots\phi(\mathbf{x}_r)>_c\; e^{-i\mathbf{p}_i\cdot\mathbf{x}_i} \quad ,$$

and the truncated r-point function is defined by

$$\tilde{\Gamma}_r(\mathbf{p}_1,\ldots,\mathbf{p}_r) = \frac{\tilde{G}_r(\mathbf{p}_1,\ldots,\mathbf{p}_r)}{\tilde{G}(\mathbf{p}_1)\tilde{G}(\mathbf{p}_2)\ldots\tilde{G}(\mathbf{p}_r)} \quad .$$

$\tilde{G}_r$ contains the denominator of the RHS as a factor and $\tilde{\Gamma}_r$ is often a more useful quantity with which to work. Then, for the 2-point function

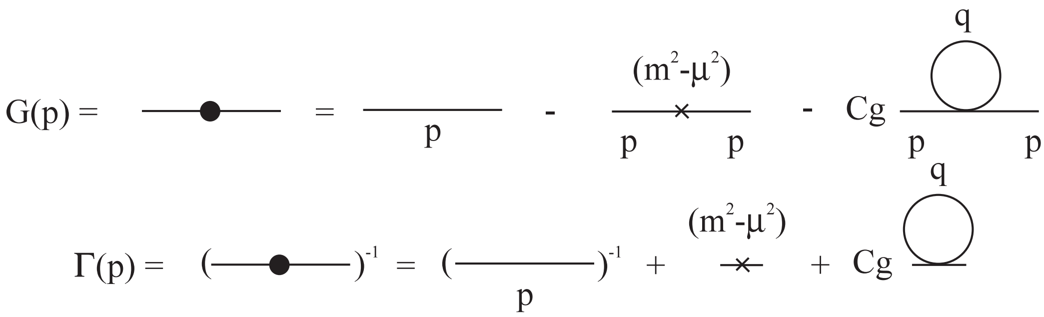
$$\tilde{\Gamma}(\mathbf{p}) \;=\; \tilde{G}^{-1}(\mathbf{p}) \;=\; \underbrace{p^2+\mu^2}_{\tilde{G}_0^{-1}(\mathbf{p})} \;+\; (m^2-\mu^2) \;+\; \frac{g}{2}\int \frac{d^Dq}{(2\pi)^D}\,\tilde{G}_0(\mathbf{q}) \;+\; \ldots O(g^2)\;.$$

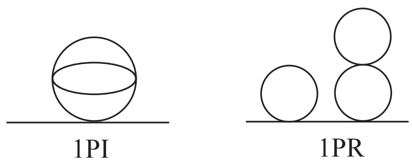
In graphical representation we have

The expansion shown for $\tilde{\Gamma}$ is exact to $O(g)$ but corresponds to a sum over selected **1-particle irreducible** graphs in $\tilde{G}$. this can be seen as follows.

$$\tilde{G} = \frac{1}{\tilde{\Gamma}} = \frac{1}{\tilde{G}_0^{-1}+\delta m^2+\Sigma} \quad ,$$

where Σ represents **1-particle irreducible** (1PI) diagrams. This means that they cannot be separated into two pieces by cutting one line only. E.g.,

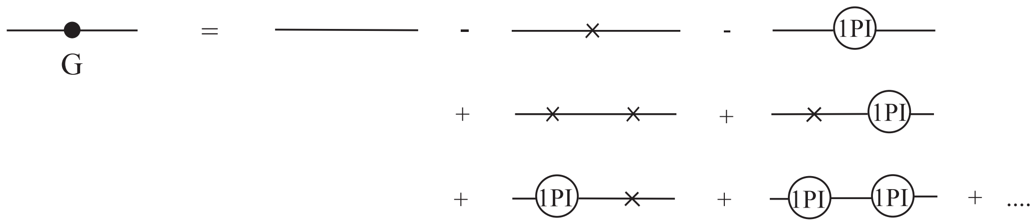


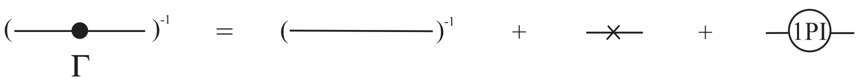


Expand the above expression to give

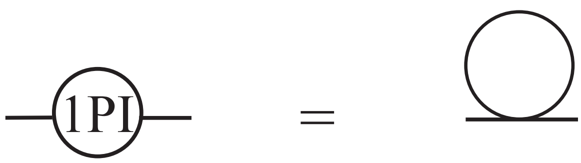
$$\begin{aligned}\tilde{G} = \tilde{G}_0 \ &- \ \tilde{G}_0\,(\delta m^2 + \Sigma)\,\tilde{G}_0 \\ &+ \ \tilde{G}_0\,(\delta m^2 + \Sigma)\,\tilde{G}_0\,(\delta m^2 + \Sigma)\,\tilde{G}_0 \ + \ldots\end{aligned}$$

Diagrammatically we have



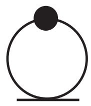


To $O(g)$:



Note that $\tilde{\Gamma}$ is **truncated** because it contains no propagator factors, $\tilde{G}$, for external legs.

The expansion for $\tilde{\Gamma}(\mathbf{p})$ can be improved by summing over all **bubble insertions** on the internal loop. Then

$\Gamma(p) =$ ()$^{-1}$ $= \; p^2 + m^2 \; + \; Cg$  $+ \; \ldots\ldots$

For our smoothed theory we have

$$m^2 \equiv m^2(\Lambda, T).$$

Remember that m^2 is not necessarily positive. Then

$$\tilde{G}(\mathbf{p}) \;=\; \frac{Z_2(\mathbf{p}^2, T)}{\mathbf{p}^2 + m^2(0,T)} \;.$$

Here $m^2(0,T) = \xi^{-2}(T)$ and is positive. The loop correction **renormalizes** m^2 additively and the resulting effective mass is positive. Now

$$\chi^{-1} \;=\; \tilde{\Gamma}(\mathbf{p}=0) = \frac{m^2(0,T)}{Z_2(0,T)} \;.$$

χ^{-1} vanishes at $T = T_c$ which is given by the vanishing of the renormalized mass:

$$\lim_{T \to T_c} m^2(0,T) = 0.$$

Note that in general $\lim_{T\to T_c} Z_2(0,T) = 0$ as well but that the numerator dominates. In fact for our present discussion

$$Z_2 = 1 + O(g^2) \;,$$

and hence to $O(g)$ we have $Z_2 = 1$.

Now

$$m^2(0,T) \;=\; m^2(\Lambda, T) \;+\; \frac{g}{2}\int^{\Lambda} \frac{d^Dp}{(2\pi)^D}\, \frac{1}{\mathbf{p}^2 + m^2(0,T)} \;+ \ldots \;.$$

Note that the bubble summation removed all reference to μ^2. In fact we can see that by choosing $\mu^2 = m^2(0,T)$ in the first place we automatically get the improved formula from the one-loop expression. The coefficient of $\frac{1}{2}$ comes from the symmetry factor as described above. Using $m^2(0,T_c) = 0$ we write

$$\begin{aligned} m^2(0,T) \;&=\; m^2(\Lambda,T) \;-\; m^2(\Lambda,T_c) \;+\; \frac{g}{2}\int^{\Lambda} \frac{d^Dp}{(2\pi)^D} \left[\frac{1}{\mathbf{p}^2 + m^2(0,T)} - \frac{1}{\mathbf{p}^2}\right] \\ &=\; At - \frac{g}{2}\, m^2(0,T) \int^{\Lambda} \frac{d^Dp}{(2\pi)^D}\, \frac{1}{(\mathbf{p}^2 + m^2(0,T))\mathbf{p}^2} \;+ \ldots \end{aligned}$$

Where $t = (T - T_c)/T_c$.

We consider two cases:

$D > 4$ The integral is dominated by large $\mathbf{p}$ and is **finite** in the limit $m^2 \to 0$. Then

$$m^2(0,T) = At + Bm^2(0,T).$$

This is consistent with the Landau behaviour

$$m^2(0,T) = Ct$$

$D < 4$ The integral is sensitive to low $\mathbf{p}$ and is **Infra-Red divergent** unless $m^2 > 0$:

$$m^2(0,T) = At + Bm^{D-2}(0,T)$$

This is clearly inconsistent with the Landau assumption: the integral **is** important. IR divergences invalidate Landau's assumption.

$D = 4$ A marginal case with IR logarithmic corrections

$$m^2(0,T) = At + Bm^2 \log m^2.$$

Get Landau behaviour modified by logs.

Hence $D_c = 4$ and for $D \leq 4$ IR effects destroy mean-field predictions.

This theory applies to an **ordinary** critical point in Landau's theory: the coefficient of ϕ^2 vanishes. If, for some reason, the ϕ^4 were absent and the interaction started at the $g\phi^6$ term then the loop contribution would be

$$\sim \int^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2(0,T)} \int^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{1}{\mathbf{q}^2 + m^2(0,T)} \quad .$$

Subtracting off the value at $m^2 = 0$ and isolating the most IR divergent integral gives

$$\int^{\Lambda} \frac{d^D p}{(2\pi)^D} \int^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{1}{\mathbf{p}^2(\mathbf{p}^2 + m^2(0,T))(\mathbf{q}^2 + m^2(0,T))} \quad .$$

There are no IR problems for $2D - 6 > 0$, i.e., $D > 3$. Hence in this case $D_c = 3$. In Landau's theory this situation corresponds to a **tricritical** point: the coefficients of ϕ^2 and ϕ^4 vanish.

In general, for interaction ϕ^{2n} we find

$$(n-1)D_c - 2n = 0,$$

or

$$D_c = \frac{2n}{n-1}.$$

This interaction is relevant only for critical points of order n. For $D = 2$ **all** critical points show anomalous behaviour but for $D = 3$ only critical and tricritical points are anomalous, and even then the corrections to mean-field predictions at the tricritical point are only logarithmic.

Notes and Extra Material

15 The Ginsberg Criterion for D_c

Mean Field Theory breaks down because the assumption that we may ignore fluctuations as $T \to T_c$, $\xi \to \infty$ is wrong. The alternative way to say this is that Infra-Red effects are important. This is true for sufficiently low dimensions $D \leq D_c$, where D_c depends on the kind of transition being considered: $D_c = 4$ for a Critical Point and $D_c = 3$ for a TriCritical Point.

The susceptibity χ per unit volume is

$$\chi \sim \int d^D x \, G(\mathbf{x}) \sim \int d^D x \frac{e^{-r/\xi}}{r^{D-2}} \sim \xi^2 .$$

In LG theory we found that $\chi \sim |t|^{-1}$ and so we deduce that $\xi^2 \sim |t|^{-1}$ as well. I.e., for LG have $[t] = 1/2$.

The fluctuations are coherent, or correlated, over a block of linear size $L \sim \xi$ so that $V \sim \xi^D$. Measure everything now "per block of size ξ"; this is the new "unit volume".

(1) Now looking at the block of size $\sim \xi$, the fluctuation is of magnitude

$$\Delta M^2 = \int d^D x d^D y \, \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle - \langle \phi(\mathbf{x}) \rangle \langle \phi(\mathbf{y}) \rangle \equiv V \int d^D x \, G(\mathbf{x}).$$

I.e., the susceptibility (up to factors). Thus, from above

$$\Delta M^2 = V \xi^2 \sim \xi^{D+2}.$$

(2) In LG theory for some general continuous transition with $T \leq T_c$ the free energy is

$$A \sim -a_2 |t| M^2 + A_{2n} M^{2n} + \ldots \implies M \sim V |t|^{1/(2n-2)}$$

Thus, we find

$$M \sim \xi^{D-1/(n-1)}.$$

Then for LG to work we need that the fluctuations become unimportant w.r.t. M as $t \to 0$. Now,

$$\frac{\Delta M^2}{M^2} \sim \frac{\xi^{(D+2)}}{\xi^{2D-2/(n-1)}}$$

and we thus require

$$(D+2) - 2D + 2/(n-1) < 0 \implies D > D_c = \frac{2n}{n-1}.$$

For $n = 2$ (CP) have $D_c = 4$, and for $n = 3$ (TCP) have $D_c = 3$.

Our anomalous scaling analysis works for $D \leq D_c$ where generally the fixed points are at non-zero values of the couplings. For $D > D_c$ Mean Field Theory holds and fixed points are Gaussian in nature. The change over requires an analysis of so-called "dangerous irrelevant variables".

16 Calculation of the Critical Index

This section is concerned with the calculation of the critical index ν in the Ising model. We work with ϕ^4 field theory in D dimensions.

$$\begin{aligned} S(\phi) &= \int d\mathbf{x}\ \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4 \ + \ldots \\ \left[m^2\right] &= 2 \qquad [g] = 4 - D \equiv \epsilon \ . \end{aligned}$$

We may imagine that we have obtained this action by integrating out the short wavelength modes. In general all types of interaction are present. Clearly ϕ has been renormalized to give the canonical form to the kinetic term. Alternatively, we can accept that $S(\phi)$ has been generated from the Ising model by the transformation described in earlier sections. We note for interest that the Ising model is identical to ϕ^4 field theory in the limit $m^2 \to -\infty,\ \ g \to \infty$ such that $6|m^2|/g = 1$.

Suppose that we have integrated over all momenta $p > \Lambda$ and that the effective parameters are $m^2(\Lambda, T)$ and $g(\Lambda, T)$. We shall also allow for multiplicative field renormalization constant $Z(\Lambda, T)$. Now integrate over the next momentum slice. Write

$$\phi(\mathbf{x}) = \phi_>(\mathbf{x}) + \phi_<(\mathbf{x}) \ .$$

$\phi_>$ contains contributions from momenta in the range $\Lambda - \delta\Lambda \le p < \Lambda$.

$\phi_<$ contains contributions from momenta in the range $0 \le p < \Lambda - \delta\Lambda$.

Then

$$\mathcal{Z} = \int d\phi_< d\phi_>\ e^{-S(\phi_> + \phi_<)} \quad .$$

We do the integrals over $\phi_>$ to get an action as a function of $\phi_<$ only.

$$\begin{aligned} S(\phi_< + \phi_>) = S(\phi_<) + S(\phi_>) \ &+ \ \int d\mathbf{x} \left[\nabla\phi_>\nabla\phi_< + m^2\phi_<\phi_>\right] \\ &+ \ \frac{g}{12}\int d\mathbf{x} \left[2\phi_<^3\phi_> + 3\phi_<^2\phi_>^2 + 2\phi_<\phi_>^3\right] \ . \end{aligned}$$

The term quadratic in the fields vanishes identically. Since in p-space $\phi_>$ and $\phi_<$ have disjoint support

$$\int d\mathbf{x}\phi_>(\mathbf{x})\phi_<(\mathbf{x}) = \int d\mathbf{p}\tilde{\phi}_<(\mathbf{p})\tilde{\phi}_>(-\mathbf{p}) = 0 \ .$$

Then

$$\mathcal{Z} = \int d\phi_<\ e^{-S(\phi_<)} \ \int d\phi_>\ e^{-S(\phi_>)}\ e^{-\frac{g}{12}\int d\mathbf{x}\left[2\phi_<^3\phi_> + 3\phi_<^2\phi_>^2 + 2\phi_<\phi_>^3\right]} \ .$$

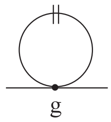
As before we expand all exponentials of non-quadratic terms

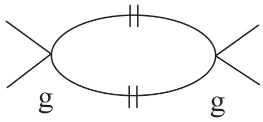
$$\begin{aligned} \mathcal{Z} \ =& \\ & \int d\phi_<\ e^{-S(\phi_<)} \ \int d\phi_>\ e^{-\frac{1}{2}(\nabla\phi_>^2 + \frac{1}{2}m^2\phi_>^2)} \cdot \left[1 - \frac{g}{4!}\phi_>^4 + \ldots\right] \cdot \\ & \left[1 - \frac{g}{12}(2\phi_<^3\phi_> + 3\phi_<^2\phi_>^2 + 2\phi_<\phi_>^3) + \frac{g^2}{288}(2\phi_<^3\phi_> + 3\phi_<^2\phi_>^2 + 2\phi_<\phi_>^3)^2 - \ldots\right] \ . \end{aligned}$$

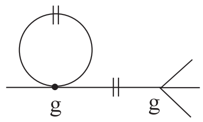
The integrals over odd functions of $\phi_<$ vanish and so will be omitted from now on.

After the $d\phi_>$ integrals have been done we are left with a **polynomial** in $\phi_<$ and its derivatives. We gather these terms up and absorb them into $S(\phi_<)$ by a redefinition or **renormalization** of the coupling constants.

The important one-loop terms and their diagrammatic representation are

$-\frac{g}{4}\int d\mathbf{x}\phi_>^2\phi_<^2$  $-\frac{1}{4}\ \phi_<^2(\mathbf{x})$

$\frac{g^2}{32}\int d\mathbf{x}\phi_>^2\phi_<^2\int d\mathbf{y}\phi_>^2\phi_<^2$ $\frac{1}{16}\phi_<^2(\mathbf{x})$  $\phi_<^2(\mathbf{y})$

$\frac{g^2}{36}\int d\mathbf{x}\phi_>\phi_<^3\int d\mathbf{y}\phi_>^3\phi_<$ $\frac{1}{12}\phi_<(\mathbf{x})$  $\phi_<^3(\mathbf{y})$

 is the propagator of $\phi_>$ and has support in p-space only for $\Lambda-\delta\Lambda \leq p < \Lambda$. The last of the terms above is zero since $\phi_<$ has disjoint p-space support from that of the intermediate propagator.

Other terms give either

(i) zero, since they contain an odd power of $\phi_>$, **or**

(ii) a six-point function and a higher order ($O(g^2)$) mass renormalization term.

We can now read off the renormalizations of $m^2(\Lambda, T)$ and $g(\Lambda, T)$ to 1-loop

$$m^2(\Lambda-\delta\Lambda, T) = m^2(\Lambda, T) + \frac{g(\Lambda, T)}{2}\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d^Dp}{(2\pi)^D}\ \frac{1}{\mathbf{p}^2+m^2(\Lambda, T)}\ . \tag{124}$$

In momentum space we have for the 4-point function

$$\frac{g^2}{16}\int\frac{d^Dq}{(2\pi)^D}\tilde{\phi}_<^2(\mathbf{q})\ \Gamma(\mathbf{q})\ \tilde{\phi}_<^2(-\mathbf{q})\ ,$$

where

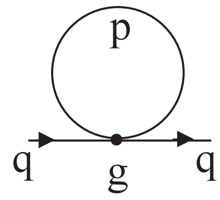
$$\Gamma(\mathbf{q}) = \int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d^Dp}{(2\pi)^D}\ \frac{1}{\mathbf{p}^2+m^2(\Lambda, T)}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2(\Lambda, T)}\ ,$$

with the restriction $\Lambda-\delta\Lambda \leq |\mathbf{p}+\mathbf{q}| < \Lambda$.

We can expand $\Gamma(\mathbf{q}) = \Gamma(0) + q^2\Gamma'(0) + \ldots$ and then the operator generated is

$$\frac{g^2}{16}\left[\Gamma(0)\int d\mathbf{x}\phi_<^4 + 4\Gamma'(0)\int d\mathbf{x}(\phi_<\nabla\phi_<)^2 + \ldots\right] .$$

Note that **new** operators are generated as well as ones we are interested in. It should be remarked at this point that the analysis of the higher dimension operators generated in this way is not easy and is better approached by other renormalization group techniques However, there does exist work that has pushed the above style of analysis, i.e., momentum thinning, successfully to the study of the role of such operators. To 1-loop we do not encounter these difficulties. Also note that because the graph



is independent of $\mathbf{q}$ we do not generate the operator $(\nabla\phi_<)^2$ to 1-loop. More on this later.

This slicing approach is different from that in the section on mass renormalization earlier. The slicing builds up the final result in stages; each diagram gives a contribution proportional to $\delta\Lambda$, and by absorbing these contributions into a redefinition of the couplings we obtain RG equations for how the coupling constants flow with Λ. The procedure corresponds to a resummation of particular contributions in the perturbation series, not unlike the equations we derived for the mass renormalization.

Then we find the equation for g

$$g(\Lambda-\delta\Lambda, T) = g(\Lambda, T) - \frac{3}{2}g^2(\Lambda, T)\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d^Dp}{(2\pi)^D}\,\frac{1}{(\mathbf{p}^2+m^2(\Lambda,T))^2} .$$

We note that from the equation for m^2, eqn. (124), the renormalization is **positive**. Hence if we want

$$\lim_{\Lambda\to 0} m^2(\Lambda, T_c) = 0 ,$$

we require $m^2(\Lambda, T) < 0$: the "bare" mass is **negative**. The expansion is still fine since it is only necessary that the gaussian exponent gives convergent integrals. Inspecting the denominator of the integrand in eqn. (124) this demands only that this denominator remains positive and so that

$$\Lambda^2 > -m^2(\Lambda, T) \qquad \forall\Lambda .$$

It can be verified in what follows that this inequality is always respected.

We now rewrite the two renormalization equations in differential form in terms of **dimensionless** quantities.

$$\begin{aligned} u^2(b,T) &= \Lambda^{-2} m^2(\Lambda,T) \\ \lambda(b,T) &= \Lambda^{-\epsilon} g(\Lambda,T)\ , \end{aligned}$$

where $b = \log(\Lambda_0/\Lambda)$ and $\epsilon = 4 - D$. Λ_0 is the cut-off for the original theory.

Then we have the evolution equations

$$\begin{aligned} \frac{du^2}{db} &= 2u^2 + \frac{\Omega_D}{2(2\pi)^D}\frac{\lambda}{1+u^2} \\ \frac{d\lambda}{db} &= \epsilon\lambda - \frac{3\Omega_D}{2(2\pi)^D}\frac{\lambda^2}{(1+u^2)^2}, \end{aligned} \tag{125}$$

Ω_D is the surface area of a unit sphere in D dimensions: $\Omega_4 = 2\pi^2$.

In order to integrate these equations the propagator for $\phi_<$ must take the canonical form $1/(\mathbf{p}^2+m^2)$. In general, after integrating over the momentum shell of thickness $\delta\Lambda$, the renormalized action, $S_R(\phi_<)$ will be of the form

$$\begin{aligned} S_R(\phi_<) &= S(\phi_<) + \frac{1}{2}\delta m^2\phi_<^2 + \frac{1}{4!}\delta g\phi_<^4 \\ &- \frac{1}{2}\delta z(\nabla\phi_<)^2 + \frac{1}{4!}\delta g'(\phi_<\nabla\phi_<)^2 + \ldots, \end{aligned}$$

where δz determines the field rescaling or renormalization. We would like to keep the coefficient of $\nabla\phi_<\nabla\phi_<$ to be unity at each step (just like in the Gaussian model) and so we can renormalize the field at each step so that this is true. This means we start the next step from the same style of action as the previous one and can so easily iterate (i.e., integtate) the RG equations. This is a component of the scaling part of the RG apprpoach which we describe now.

As in the Gaussian model, we can perform a **rescaling** so that we recover the **same** cut-off Λ_0 but in new units. Note, that under this rescaling u and λ do not change since they are dimensionless. We carry out the rescaling every time we integrate out a shell from Λ to $\Lambda - \delta\Lambda$. Of course, the value of b keeps track of the total amount of rescaling.

Let $\kappa = e^b$. After integration of the RG equation above we can write the RG followed by the rescaling step as follows:

In momentum space we have

$$\begin{aligned} \Lambda_0 \ &\to\ \Lambda\ \to\ \Lambda_0\ =\ \kappa\,\Lambda\ , \\ \mathbf{p}\ &\to\ \mathbf{p}\ \to\ \mathbf{q}\ =\ \kappa\,\mathbf{p}\quad \Rightarrow\quad 0 \le q \le \Lambda_0\ , \\ \tilde\phi_<(\mathbf{p})\ &\to\ \tilde\phi_<(\mathbf{p})\ \to\ \tilde\phi'(\mathbf{q})\ =\ \tilde Z^{-1}\,\tilde\phi_<(\mathbf{p})\ . \end{aligned} \tag{126}$$

In coordinate space we have

$$\begin{aligned} a_0\ &\to\ a\ \to\ a_0\ =\ a/\kappa\ , \\ \mathbf{x}\ &\to\ \mathbf{x}\ \to\ \mathbf{x}_0\ =\ \mathbf{x}/\kappa\ , \\ \phi_<(\mathbf{x})\ &\to\ \phi_<(\mathbf{x})\ \to\ \phi'(\mathbf{x}_0)\ =\ Z^{-1}\,\phi_<(\mathbf{x})\ . \end{aligned} \tag{127}$$

Since $\tilde{\phi}(\mathbf{p})$ is the Fourier transform of $\phi(\mathbf{x})$ we find immediately that $\tilde{Z}(b) = \kappa^D Z(b)$. This follows because the measures in the Fourier transforms for $\mathbf{p}$ and $\mathbf{q}$ are $d^D p$ and $d^D q$, respectively, and $d^D q = \kappa^D d^D p$; it is just like in the Gaussian model.

From the definitions of u and λ we have, in either case,

$$\begin{aligned} u(b,T) &\rightarrow u_R(b,T) = Z(b,T)u(b,T) \\ \lambda(b,T) &\rightarrow \lambda_R(b,T) = Z^4(b,T)\lambda(b,T) \ , \end{aligned}$$

and the RG equations above must be modified accordingly to give the flows for u_R and λ_R.

However, we have note that to the order in which we are working,(1-loop), $Z = 1$. Henceforth, we set $Z = 1$ above, and identify (u_R, λ_R) with (u, λ). In a general case, we **would** need to compute $Z(b,T)$ but in ϕ^4 theory this is hard and needs some further work.

It is important to remark at this stage that the renormalization choices I am making (e.g. defining the coefficient of the kinetic term to be unity), are not forced on me by physics: they are convenient for the perturbative-style analysis. Other choices may be necessary in more complex situations in order to reveal the structure of more complex phase transitions in the most effective way.

The transformation takes the form

$$\begin{aligned} S(\phi, u^2, \lambda_n, \Lambda_0) &\overset{\text{integration}}{\longrightarrow} S(Z^{-1}\phi, u^2(b), \lambda_n(b), e^{-b}\Lambda_0) \\ &\overset{\text{rescaling}}{\longrightarrow} S(\phi, u_R^2(b), \lambda_{Rn}(b), \Lambda_0) \ . \end{aligned}$$

Here $Z \equiv Z(b)$, $u = m/\Lambda$ and λ_n is the dimensionless coupling derived from the actual coupling g_n associated with field monomial of order n. This is the core of the renormalization group transformation with the RG flow equations given above in (125). Note that by dealing with **dimensionless quantities** all explicit reference to the cut-off Λ disappears.

The fixed points (u^{*2}, λ^*) of these RG equations are

$$\begin{array}{llll} \epsilon < 0 & (0 \ , \ 0) & \text{trivial f.p. - IR attractive} \\ \epsilon > 0 & (0 \ , \ 0) & \text{trivial f.p. - UV attractive} \\ & (-\dfrac{\epsilon}{6} \ , \ \dfrac{16\pi^2}{3}\epsilon) & \text{non-trivial f.p. - IR attractive} \end{array}$$

In the neighbourhood of the non-trivial f.p. we have, to $O(\epsilon)$

$$\begin{pmatrix} dx/db \\ dy/db \end{pmatrix} = \begin{pmatrix} 2 - \frac{1}{3}\epsilon & \frac{1}{3}\epsilon \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ,$$

where $x = u^2 - u^{*2}$, $y = \lambda - \lambda^*$. As before $x \propto t$. then

(1) λ and hence g is an **irrelevant** variable and λ is driven by the RG flow to the non-trivial fixed point. Thus **whatever** coupling λ_0 we start with in the original theory we **always** end up with the unique value $\lambda^* = (16\pi^2/3)\epsilon$.

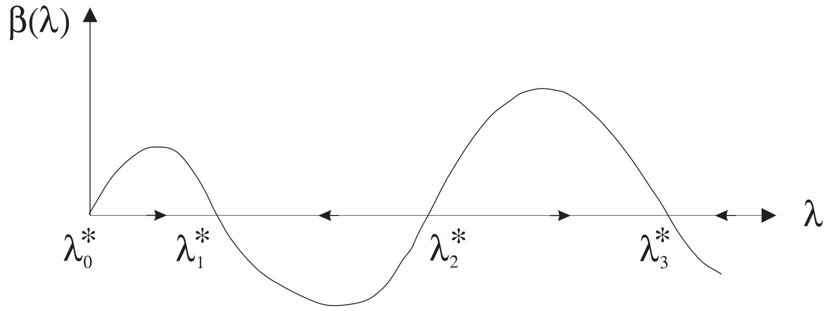
(2) in contrast, u^2 is a relevant variable and so we know that t must be tuned to zero so that u^2 (and hence m^2) lies in the critical surface. From above we see that

$$\begin{aligned} \lambda_t &= 2-\frac{1}{3}\epsilon \quad\Rightarrow\quad \nu = \frac{1}{\lambda_t} = \frac{1}{2}+\frac{1}{12}\epsilon \\ \alpha &= 2-D\nu \quad\Rightarrow\quad \alpha = \frac{5}{6}\epsilon\ . \end{aligned}$$

(3) Consider a flow equation of the form

$$\frac{d\lambda}{db} = \beta(\lambda)\ .$$

This equation has fixed points λ_p^* where $\beta(\lambda_p^*) = 0$, e.g.,



If $\lambda_0^* < \lambda(0,T) < \lambda_2^*$ then $\lim_{b\to\infty}\lambda(b,T) = \lambda_1^*$

If $\lambda_2^* < \lambda(0,T) < \lambda_4^*$ then $\lim_{b\to\infty}\lambda(b,T) = \lambda_3^*$

Fixed points λ_p^* where $\beta'(\lambda_p^*) < 0$ are **Infra-Red** attractive, and alternatively if $\beta'(\lambda_p^*) > 0$ the points are **Ultra-Violet** attractive.

Note that

(a) In **statistical mechanics** we are **given** the UV or **bare** coupling $\lambda(0,T)$, and the IR or **renormalized** coupling is an IR point of $\beta(\lambda)$.

(b) In **quantum field theory** we are **given** the IR or **renormalized** coupling $\lambda(\infty,T)$: it is a renormalization condition from experiment or other **low energy** condition. Since $\lambda(\infty,T)$ is not generally a fixed point of $\beta(\lambda)$ it must be that the UV or **bare** coupling is a UV fixed point of β. (It should be noted that although we have worked exclusively in Euclidean space it is believed that the divergences are the same as for the Minkowski version of the theory. This belief can be demonstrated in perturbation theory.)

In our calculation we found the fixed points (u^{*2},λ^*)

$\epsilon < 0$	$(0\ ,\ 0)$	trivial f.p. - IR attractive
$\epsilon > 0$	$(0\ ,\ 0)$	trivial f.p. - UV attractive
	$(-\frac{\epsilon}{6}\ ,\ \frac{16\pi^2}{3}\epsilon)$	non-trivial f.p. - IR attractive

It is believed that there are no others, not even at $\lambda = \infty$. This means for quantum field theory that there is no UV fixed point to be associated with the bare coupling **except** the one at $(0, 0)$ when $\epsilon < 0$. Hence, it follow that

$$\begin{array}{lll} \epsilon < 0 & \lambda_R & = 0 \\ \epsilon \geq 0 & \lambda_R & \leq \frac{16\pi^2}{3}\epsilon, \end{array}$$

where $\lambda^R = \lambda(b = \infty, T)$. This is the statement of triviality for ϕ^4 quantum field theory. It means that in dimension four or greater the only consistent renormalized ϕ^4 theory in the limit of infinite cut-off is a **free** theory: $\lambda_R = 0$. Of course, we are not required to take the cut-off to infinity, rather we should keep it much larger than the physical momentum scales on which the theory is being applied. Even then, for **finite** Λ the analysis presented gives an upper limit to λ_R. Such upper limits have been used to set upper bounds on the Higgs mass in the most common version of the standard model.

Since ϵ is small we can get a reliable expansion in ϵ for both u^{*2} and λ^*. This is the epsilon expansion

$$\lambda^* = \sum_{n=1}^{\infty} a_n \epsilon^n \ .$$

It is hoped that the radius of convergence is greater than one, thus including results for $D = 3$.

(4) In the Landau theory we associate the coefficient of M^4 with the **low-momentum** dimensionful coupling

$$g(T) = \Lambda_0^\epsilon \lim_{b \to \infty} \ e^{-\epsilon b} \lambda(b, T) \ .$$

This is because $g(T)$ is the coefficient of M^4 in the **effective** action derived by the Legendre transform. ($Z = 1$ to $O(\epsilon)$ otherwise we would need to include Z^2 in the definition of $g(T)$).

Let

$$g(b, T) = e^{-\epsilon b} \lambda(b, T) \ .$$

Then the equation for the flow of λ in the neighbourhood of the fixed point implies

$$\frac{dg}{db} = -\frac{3}{16\pi^2} \ e^{\epsilon b} g^2 \ ,$$

or

$$\frac{1}{g(\bar{b})} - \frac{1}{g(b)} = -\frac{3}{16\pi^2} \ e^{\epsilon \bar{b}} \left(e^{\epsilon(b - \bar{b})} - 1 \right) \ ,$$

and thus

$$\begin{aligned} g(b) &= \frac{g(\bar{b})}{1 + \frac{a(\epsilon)}{\epsilon} \left(e^{\epsilon(b - \bar{b})} - 1 \right)} \ . \\ a(\epsilon) &= \frac{3}{16\pi^2} \ e^{\epsilon \bar{b}} \ . \end{aligned}$$

Denoting $g(T) \equiv g(\infty, T)$ we have the following results

$\epsilon > 0$: $g(T) = 0$. Hence Landau's method fails and the relevant physics is encoded in **how** $g(b, T) \to 0$ as $b \to \infty$.

$\epsilon < 0$: We have

$$g(T) = \frac{g(\bar{b})}{1 + \frac{a(\epsilon)}{|\epsilon|}} \; .$$

Since here $g(T)$ is finite and non-zero Landau's method works. As before we are assuming that $\bar{b}$ is large enough so that the equations we integrated to get this result are valid. In this case $g(T)$ is actually independent of $\bar{b}$.

$\epsilon = 0$: $g(T) = 0$. Landau theory fails but

$$\begin{aligned} g(b) &= \frac{g(\bar{b})}{1 + \frac{3}{16\pi^2}(b - \bar{b})} \\ g(b) &= \frac{g(\bar{b})}{1 + \frac{3}{16\pi^2}\log\frac{\bar{\Lambda}}{\Lambda}} \; . \end{aligned}$$

I.e., only logarithmic violations of Landau theory.

17 General Ideas

Note. The general ideas of relating physical phenomena on widely differing scales by renormalization group methods is widely applicable in many fields. For example:

(i) general diffusion models such as diffusion on fractal structures and the large scale effects of diffusive transport processes in fluid flow;

(ii) turbulence in fluids. The fluid velocity field, $\mathbf{u}(\mathbf{x}, t)$, of fully developed turbulence has energy density, as a function of wavenumber $\mathbf{k}$,

$$k^{-2}E(\mathbf{k}) \sim \frac{1}{V}\int d\mathbf{x}d\mathbf{y}\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{u}(\mathbf{y}, t)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

with $E(\mathbf{k}) \sim k^{-\frac{5}{3}}$ for large k.

This result is derived by "naive" dimensional analysis (Kolmogorov). However, a full solution of the Navier-Stokes equation can correct the exponent:

$$E(\mathbf{k}) \sim k^{-\frac{5}{3}-\eta},$$

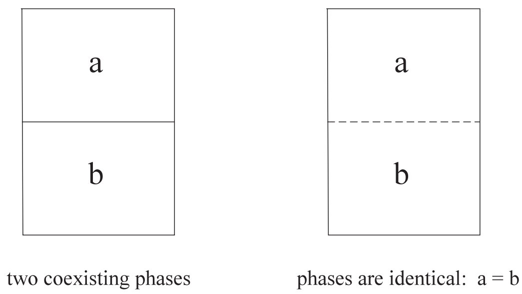
where η is the anomalous term. However, recent work suggests that normally $\eta = 0$

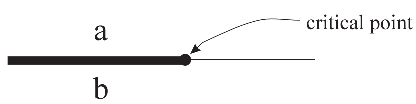
17.1 Domains and the Maxwell Construction

To realise a value of M between the values $\pm M_P$ associated with the pure system boundaries between phases or domains form. In each domain the magnetisation is oriented differently and so the bulk average magnetisation can be any value in the range $-|M|$ to $|M|$, where $|M|$ is the magnetization of a pure domain. The walls do increase the energy of the system by $\Delta\epsilon$ and there is also an increase in the entropy, ΔS. since there are many ways of realising the mixed state. However, the resultant change in the free-energy, $\Delta F = \Delta\epsilon - T\Delta S$, depends on the surface area of the walls and is negligible in the limit of very large volume. Of course, the actual way in which domains, or bubbles, form and move is very important (e.g. in the early universe, cosmic string formation etc.) but needs more analysis than the embodied in the Maxwell construction.

17.2 Types of critical point

An ordinary **critical point** has $\kappa = 2$ and occurs when two coexisting phases become identical:



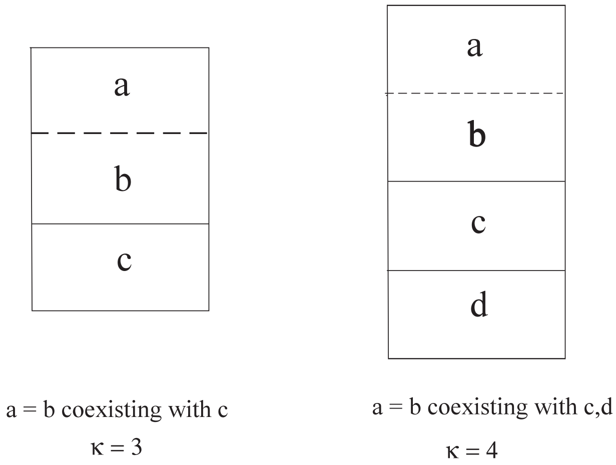


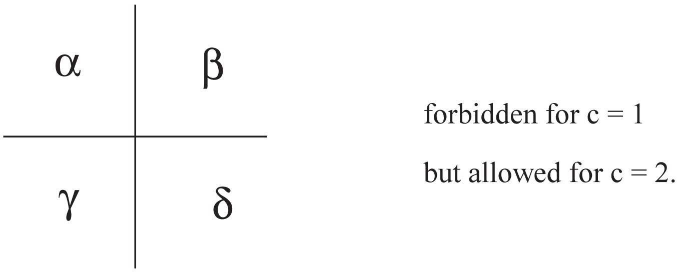
A **critical end-point** occurs with codimension κ when two coexisting phases become identical in the presence of $(\kappa - 2)$ other phases:

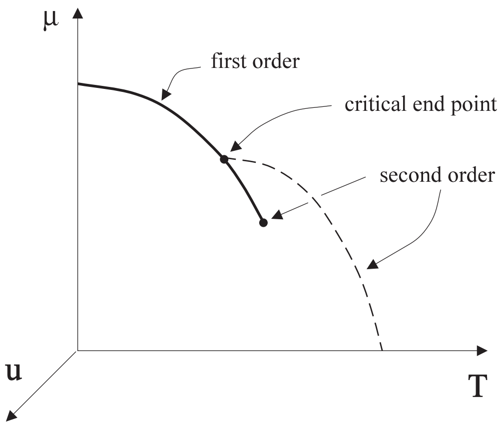
Thus the critical end-point terminates a line of critical points and also terminates a line of triple points.

Note that a phase diagram can often only be properly understood if plotted in the space of all relevant parameters. E.g., the Gibbs rule might seem to be violated since too many phases are coexisting at one point. However, if the space is enlarged in dimension this will be seen as a special case which only occurs for a particular cross-section of the enlarged space;

For the tri-critical point, a less special 2D cross-section of the same model will be:







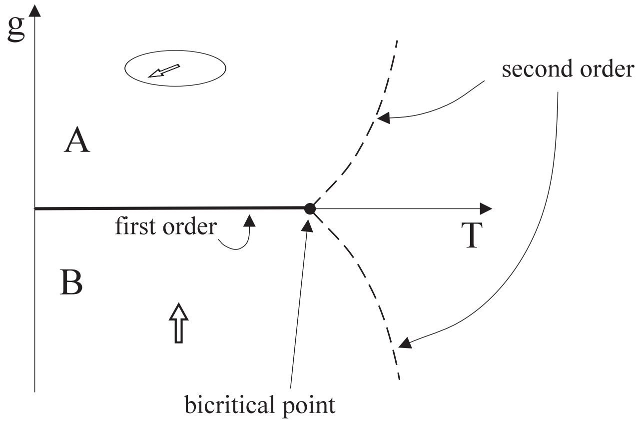
Here we have set $h = 0$ (see 3D plot) but have changed the value of a fourth parameter u.

A **bicritical point** is a critical point at which two critical lines terminate. A typical phase diagram is: A model which has a phase diagram like this is given by the Hamiltonian

$$H = -J \sum_{<ij>} \mathbf{s}_i \cdot \mathbf{s}_j + \frac{1}{2} g \sum_i \Big((\mathbf{s}_i^z)^2 - \frac{1}{2}((\mathbf{s}_i^x)^2 + (\mathbf{s}_i^y)^2) \Big)$$

$< ij >$ means nearest neighbour pairs, i.e., it labels the links on the lattice.

$\mathbf{s}_i$ is a vector at the i-th site with $|\mathbf{s}_i| = 1$.



For **low** T thermal fluctuations can be ignored and it is safe to just find the configuration (i.e., set of values) of spins $\{\mathbf{s}_i\}$ which minimizes H. Since $J > 0$ the first term causes the spins to align with each other to give ferromagnetic ordering.

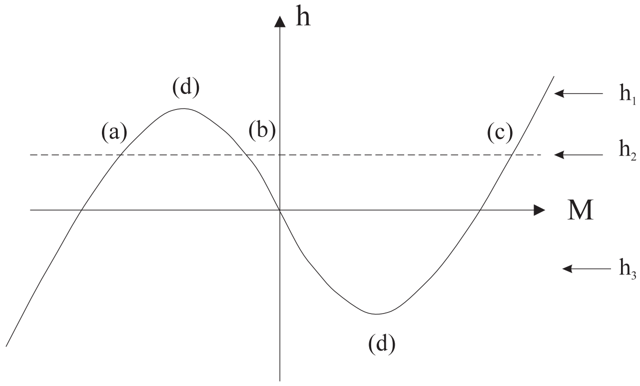
$g < 0$	Ordering is preferred along z-axis. This is phase B.
$g > 0$	Ordering is preferred in the xy-plane $\perp$ z-axis. This is phase A.
$g = 0$	Neither A nor B preferred: two-phase coexistence.

For high T ordering is absent: it is destroyed by thermal fluctuations. For $g \neq 0$ as T increases we must have a second order transition from ferromagnetic to paramagnetic phases. The surfaces to the low-T side of the lines of critical points are first-order surfaces. This can be seen by imposing a magnetic field $\mathbf{h}$ on the system with components both $\|$ and $\perp$ to the z-axis and adding the magnitude, h, of $\mathbf{h}$ (including sign) as a third orthogonal axis to generate a 3D phase plot of which our 2D plot is the $h = 0$ cross-section. Then as h changes sign the magnetisation, $\mathbf{M}$, changes discontinuously at $h = 0$. This occurs as the surfaces in our 2D phase diagram on the low-T side of the critical lines are punctured, and hence they are in fact first-order surfaces. Of course, because the order parameter is a vector the possible patterns of behaviour and the competition between the effects of the terms governed by the coupling, g, and by h is, in general, complicated. An r-critical point is where r critical lines terminate.

A **critical point** of n-th order has $\kappa = n + 2$ and is complicated.

17.3 Hysteresis

If h is tuned from positive to negative the true minimum describing equilibrium changes from one to the other, **but** it takes time to re-establish th equilibrium especially if the intervening barrier is high. Consequently, the system can be in a



metastable state corresponding to the local but not global minimum. This is the phenomenon of **hysteresis**. What happens is clear from the equation of state:

For h_1 there is unique minimum and the state is stable.

For h_2 there are two minima (a) and (c) and a maximum (b). State (c) is **stable** and (a) is **metastable**, but (b) is **unstable** corresponding to a **maximum** of A and thermodynamic inequalities are violated here.

17.4 Mean Field Theory

The mean field approach is a type of variational principle. It can be formulated in a way due to Feynman and Peierls using the result that if $g(x)$ is a convex function then $\langle g(x) \rangle \geq g(\langle x \rangle)$ whatever the probability distribution defining $\langle . \rangle$ (See Binney et al). In particular, the exponential function is convex. Let H be the Hamiltonian of a model of interest and H_0 the Hamiltonian of a related soluble model. Then we can write

$$\begin{aligned} e^{-H} &= e^{-H_0} e^{-(H-H_0)} \\ \Rightarrow & \\ e^{-F} &= \sum_\sigma e^{-H} = e^{-F_0} \sum_\sigma \underbrace{\frac{e^{-H_0}}{e^{-F_0}}}_{p_0(\sigma)} e^{-(H-H_0)} \\ \Rightarrow & \\ e^{-F} &= e^{-F_0} \left\langle e^{-(H-H_0)} \right\rangle \geq e^{-F_0} e^{-\langle H-H_0 \rangle_0} \\ \Rightarrow & \\ F &\leq F_0 + \langle H - H_0 \rangle_0 , \end{aligned} \tag{128}$$

where $\langle\rangle_0$ signifies averaging over the soluble model described by H_0. In general, H_0 will depend on a set of variational parameters, for example, the mean field magnetization M. The best estimate for F is then to minimize the RHS of (128) w.r.t these parameters. Applied to the Ising model we recover the same results as above. However, this formulation of the variational principle is very general and is used widely.

(2) For the Gaussian, or free, field theory in $D > 1$ we have $\zeta = [\sigma] = -(D-2)/2$ since the kinetic term in the Hamiltonian

$$\frac{1}{2}\int d^D x \, (\nabla\sigma(\mathbf{x}))^2 \, , \tag{129}$$

is dimensionless up to energy units. For $r \ll \xi$ there is no other dimensionful length but r and so we have (see Eq. 73)

$$\begin{aligned} G(b^{-1}r) &= G(r') = \langle \sigma_{r'} \, \sigma_0 \rangle = b^{-2\zeta} \langle \sigma_r \, \sigma_0 \rangle = b^{-2\zeta} G(r) \\ \Rightarrow \quad G(r) &\sim \frac{1}{r^{-2\zeta}} = \frac{1}{r^{D-2}} \, . \end{aligned}$$