

# Highly-symmetric travelling waves in pipe flow

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The recent theoretical discovery of finite-amplitude travelling waves in pipe flow has re-ignited interest in the transitional phenomena that Osborne Reynolds studied 125 years ago. Despite all being unstable, these waves are providing fresh insight into the flow dynamics. We describe two new classes of travelling wave which, while possessing more restrictive symmetries than previously found travelling waves of Faisst & Eckhardt (2003) and Wedin & Kerswell (2004), seem to be more fundamental to the hierarchy of exact solutions. They exhibit much higher wall-shear stresses and appear at notably lower Reynolds numbers. The first M-class comprises of the various discrete-rotationally symmetric analogues of the mirror-symmetric wave found in Pringle & Kerswell (2007) and have a distinctive double layer structure of fast and slow streaks across the pipe radius. The second N-class has the more familiar separation of fast streaks to the exterior and slow streaks to the interior and looks like the precursor to the class of non-mirror-symmetric waves already known.

**Keywords:** Pipe flow; travelling waves; turbulence transition

## 1. Introduction

The stability of the flow of a Newtonian fluid such as water through a straight pipe of constant circular cross-section has fascinated scientists ever since Reynolds' (1883) original experiments. Reynolds realised that there was a single non-dimensional control parameter for the flow,  $Re := UD/\nu$  (where  $U$  is the mean velocity,  $D$  is the pipe diameter, and  $\nu$  is the kinematic viscosity of the fluid), but that there was no unique value of this parameter for transition to occur. Instead, he noticed that the value of  $Re$  required for transition depended critically on how cleanly he performed the experiment, that is, on the ambient level of noise. Later theoretical and computational work confirmed that the steady, unidirectional, parabolic, laminar 'Hagen-Poiseuille' flow (Hagen 1839, Poiseuille 1840) realised at low  $Re$  is linearly stable at least to  $Re = 10^7$  (Meseguer & Trefethen 2003) and that pipe flow transition must be a finite-amplitude process. The variability in the critical  $Re$  across experimental realisations has indicated that this process is also sensitive to the exact form of disturbances present in the flow (e.g. Peixinho & Mullin 2007).

Once triggered, transition is generally abrupt and can lead to both localised and global forms of turbulence depending on  $Re$ . For  $1760 \lesssim Re \lesssim 2,300$ , turbulence is localised into 'puffs' (Wynanski & Champagne 1973) which have a weak front, a sharp trailing edge, and maintain their length at about  $20D$  as they propagate along the pipe. For  $2,300 \lesssim Re$ , the turbulence delocalises into 'slug' turbulence

(Wygnanski *et al* 1975) which has sharp front travelling faster than the mean flow and a trailing edge travelling slower so that it expands while propagating downstream. As a result of this varied behaviour and the sensitivity of the transition process, pipe flow has become *the* canonical example of shear flow transition.

From a dynamical systems viewpoint, the existence of turbulence requires the presence of simpler solutions of the governing equations to provide a framework in state space to sustain turbulent trajectories from simply relaminarising. Natural questions are then whether the emergence of such alternative solutions can be used to predict the threshold (lowest)  $Re$  for transition and what role they play in the process as well as the final turbulent state. Steady states and travelling waves (TWs) - fixed points in an appropriately chosen Galilean frame - are the simplest solutions.

The first step towards identifying such states was taken by Smith & Bodonyi (1982) who predicted the existence of helical modes (with swirl) through a critical layer analysis. Landman (1990), however, failed to find any numerical evidence for their existence. In the same year, Nagata (1990) used a homotopy approach to find finite amplitude solutions in plane Couette flow starting from the linearly-unstable rotating situation. A similar attempt using rotating pipe flow, however, failed to reach the vanishing rotation limit (Barnes & Kerswell 2000). By this time a more systematic approach was emerging based upon the idea of a self-sustaining process. Hamilton *et al.* (1995) suggested that streamwise rolls could create streamwise streaks which in turn could reinforce the original rolls by some unspecified non-linear process. Waleffe (1997,1998) closed the loop by showing that the streaks had inflectional instabilities which led to what he described as ‘wriggling’ in such a manner that energy is fed back into the initial rolls. He then identified such solutions in plane-Couette flow using a constructive homotopic approach. Faisst & Eckhardt (2003) and Wedin & Kerswell (2004) employed this technique within the geometry of pipe flow to find the expected travelling waves although all were without swirl and therefore unrelated to the predictions of Smith & Bodonyi (1982). Experimental observations (Hof *et al.* 2004,2005) and numerical computations (Kerswell & Tutty 2007, Schneider *et al.* 2007a, Willis & Kerswell 2008) then followed which indicated that these waves are realised albeit transiently as coherent structures in turbulent pipe flow. These waves are all born in saddle node bifurcations with the lowest at  $Re = 1251$  for TWs with a discrete rotational symmetry. Pringle & Kerswell (2007) substantially lowered this to 773 by finding travelling waves with no rotational symmetry. The  $Re$  gap between when alternative solutions exist and when sustained turbulence occurs is then certainly positive but also too large to be a useful predictor.

Recently Duguet *et al* (2008a) have adopted an entirely different approach to finding TWs based upon identifying episodes in time-dependent pipe flow which exhibit temporal periodicity. To generate a long time signal for this search, sustained flow dynamics away from the laminar state is required. The obvious source for this is studying flow on the turbulent attractor but they realised that there was a more promising alternative where the flow is less energetic and periodic episodes are more clearly evident: the laminar-turbulent boundary. This is the set of flows which separates initial conditions which immediately relaminarise from those which undergo a turbulent episode. An initial condition chosen on this boundary or ‘edge’ (Skufca *et al.* 2006) will evolve in time forever treading a tight rope between relaminarising and becoming turbulent (Toh & Itano 1999, Itano & Toh 2001, Schneider

et al 2007b). Duguet et al (2008a) identified 3 new TWs by using velocity fields from near-time periodic episodes in this energetically intermediate flow dynamics as starting guesses for an iterative Newton-Krylov solver.

The purpose of this paper is to show how the TWs found by Duguet et al (2008a) led to the discovery of 2 new *classes* of mirror-symmetric TWs hereafter referred to as the M- and N-classes, and a distinctly new family of mirror-symmetric TWs. Each class is partitioned by the discrete rotational symmetry (see (2.1) below) enjoyed by the waves which defines a *family* within the class. This suggests a natural labelling system where, for example, the family of M-class TWs with  $m$  (integer) degree of discrete rotational symmetry will be referred to as the  $Mm$  family. We argue that the mirror-symmetric wave of Pringle & Kerswell (2007) is actually part of the first family (M1) of the M-class and ‘A3’ in the nomenclature of Duguet et al (2008a) is a member of the second family (M2). We further report the families M3, M4, M5 and M6 which, unlike anything seen so far, all have a double layer structure of fast and slow streaks across the pipe radius. The second N-class of TWs has ‘C3’ in Duguet et al (2008a) as part of its second family (N2) with the families N1, N3, N4 and N5 described here for the first time. In these waves, the fast streaks are positioned near the pipe wall and slow streaks in the interior, a separation familiar from the original non-mirror symmetric TWs found by Faisst & Eckhardt (2003) and Wedin & Kerswell (2004). Since these waves only have shift-&-reflect symmetry - see (2.2) below - we refer to this original set as the S-class hereafter for convenience.

The plan of the paper is to start by discussing the various symmetries of the TWs which serve to categorize them in section 2. Beyond these, all the TWs are parametrised by their axial wavenumber  $\alpha$  which is a continuous variable with a finite range at a given  $Re$ . These waves are exact solutions of the Navier-Stokes equations when periodic flow conditions are imposed across the pipe with a spatial period equal to an integer multiple of  $2\pi/\alpha$ . The 3 TWs found recently by Duguet et al (2008a) are then introduced in section 3. Numerical codes developed in Wedin & Kerswell (2004) are used to trace out these solutions of the Navier-Stokes equations which are steady in a Galilean frame moving at a phase velocity of  $C\hat{\mathbf{z}}$ ,

$$-C \frac{\partial \mathbf{u}}{\partial z} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \frac{1}{2} \Lambda \hat{\mathbf{z}} + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad (1.1)$$

through parameter space. Here the mean velocity  $U$  and diameter  $D$  have been used to non-dimensionalise the system,  $\Lambda$  is the friction factor measuring the imposed pressure gradient,  $p$  is strictly periodic and the usual cylindrical coordinates  $(s, \phi, z)$  and corresponding velocity components  $(u, v, w)$  are adopted throughout. The numerical representation of the TWs is triply spectral with Fourier modes in the two periodic directions (axial and azimuthal) and Chebyshev polynomials in the radial direction (see Wedin & Kerswell 2004 for details). A new TW classification is then introduced in section 4 stimulated by the discoveries of many related families. A final discussion follows in section 5.

## 2. Travelling wave symmetries

The original work by Faisst & Eckhardt (2003) and Wedin & Kerswell (2004) used the homotopy approach developed by Waleffe (1997) to look for exact steady solutions in an appropriately chosen Galilean frame. In this, an artificial body forcing

is added to the Navier-Stokes equations in order to drive streamwise rolls. These advect the underlying shear towards and away from the pipe wall, thereby generating positive and negative streamwise anomalies in the mean flow called ‘streaks’. If these streaks are of sufficient amplitude they become inflexionally unstable to streamwise-dependent flows which if allowed to grow to sufficient amplitude can replace the artificial forcing as the energy source for the rolls. By choosing the forcing functions to have certain discrete rotational symmetries defined by

$$\mathbf{R}_m : (u, v, w, p)(s, \phi, z) \rightarrow (u, v, w, p)(s, \phi + 2\pi/m, z), \quad (2.1)$$

and selecting streamwise-dependent instabilities having the shift-&-reflect symmetry

$$\mathbf{S} : (u, v, w, p)(s, \phi, z) \rightarrow (u, -v, w, p)(s, -\phi, z + \pi/\alpha), \quad (2.2)$$

where  $2\pi/\alpha$  is the wavelength, exact TWs were found by continuing back to zero forcing in  $m = 2, 3, 4, 5$  and 6 discrete rotational symmetry subspaces. All appear through saddle node bifurcations with the lowest at  $Re = 1251$  corresponding to a  $\mathbf{R}_3$  symmetric TW. Each TW family is parameterised by its axial wavenumber  $\alpha$  which, at a given  $Re$ , has a finite range (e.g see figures 1 & 2 of Kerswell & Tutty 2007).

Later, motivated by a chaotic flow state found on the laminar-turbulent boundary by Schneider et al (2007b), Pringle and Kerswell (2007) generalised the forcing function to have no rotational symmetry (see also Mellibovsky & Meseguer 2007). As a result, they found an asymmetric TW (formally the missing  $\mathbf{R}_1$  state possessing one pair of streamwise rolls) with  $\mathbf{S}$  symmetry. This TW originates through a symmetry-breaking bifurcation from a state which satisfies the additional shift-&-rotate symmetry

$$\mathbf{\Omega}_m : (u, v, w, p)(s, \phi, z) \rightarrow (u, v, w, p)(s, \phi + \pi/m, z + \pi/\alpha) \quad (2.3)$$

with  $m = 1$  and is therefore mirror-symmetric about  $\phi = \pi/2$  (the plane  $\phi = 0$  being set by the shift-&-reflect symmetry). For general  $m$ ,

$$\mathbf{S}\mathbf{\Omega}_m^j = \mathbf{Z}_{\frac{j\pi}{2m}} \quad j = 1, 3, 5, \dots, 2m - 1 \quad (2.4)$$

where  $\mathbf{\Omega}_m^j$  implies  $\mathbf{\Omega}_m$  is applied  $j$  times and

$$\mathbf{Z}_\psi : (u, v, w, p)(s, \phi, z) \rightarrow (u, -v, w, p)(s, 2\psi - \phi, z) \quad (2.5)$$

represents reflection in the plane  $\phi = \psi$ . Helical generalisations, which satisfy a modified shift-&-rotate symmetry

$$\mathbf{\Omega}_1^\beta : (u, v, w, p)(s, \phi, z) \rightarrow (u, v, w, p)(s, \phi + (1 + \frac{\beta}{\alpha})\pi, z + \pi/\alpha) \quad (2.6)$$

(where  $\beta$  is the helicity, Pringle & Kerswell 2007) but no  $\mathbf{S}$ -symmetry, were also found but always at higher  $Re$  indicating that the flow prefers the streaks to be aligned with the flow rather than twisted around it.

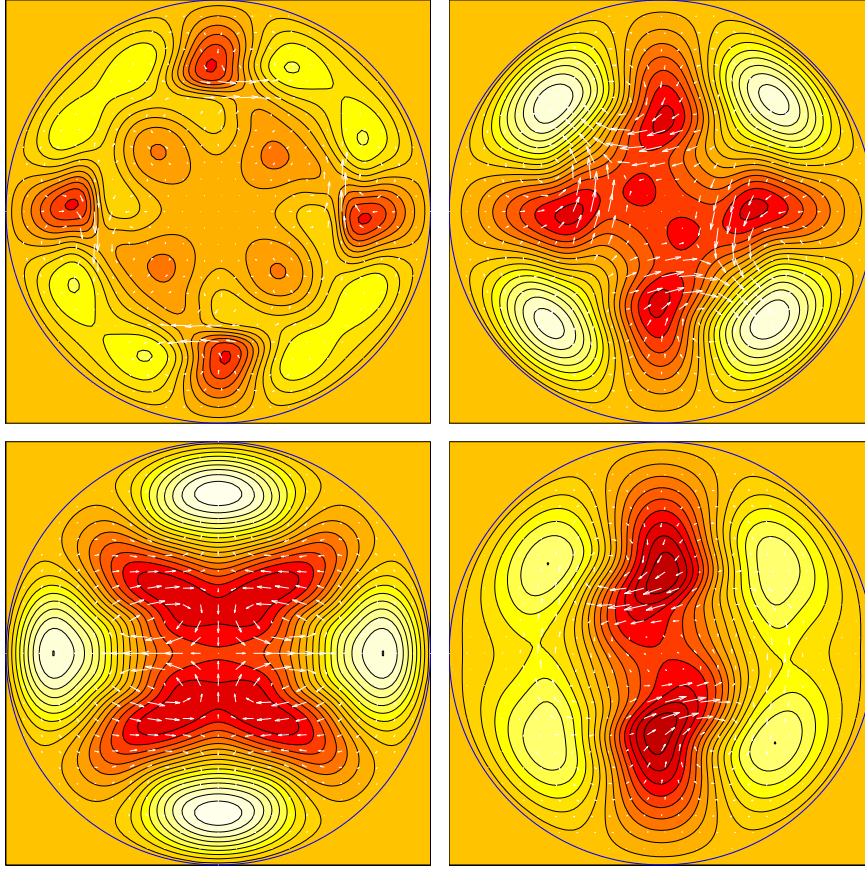


Figure 1. The travelling waves A3 (top left), C3 (top right) and Z2 (bottom left). S2 (bottom right) is also included for comparison purposes (2b.1.25 in Kerswell & Tutty 2007). All are shown at their respective values of  $\alpha^*$  (see Table 1) and at  $Re = 2400$  except for Z2 which is shown at  $Re = 3860$ . The contours indicate the magnitude of the streamwise velocity difference from the underlying laminar flow. The contours and colouring is standardised across the plots (and more generally figures 3, 4 and 10) with contours increments of  $0.02U$  running from  $-0.19U$  (dark red) to  $0.19U$  (white) (the colour outside of the pipes indicates zero). Arrows indicate the cross-stream velocities.

### 3. Travelling waves found on the laminar-turbulent boundary

The new TWs discovered by Duguet et al (2008a) were found by studying the flow dynamics on the laminar-turbulent boundary *within* the  $\mathbf{R}_2$ -symmetric subspace. Surprisingly, at  $Re = 2875$  and periodic pipe length  $5D$ , the rotationally-unrestricted situation only ever revealed the already-known asymmetric TW (Pringle & Kerswell 2007) and a weakly rotating version of it ( $A'$  in Duguet et al 2008a). The fact that new  $\mathbf{R}_2$ -symmetric TWs were observed as transient coherent structures only in the  $\mathbf{R}_2$ -symmetric subspace calculations is due to the reduced number of unstable directions they have as non  $\mathbf{R}_2$ -symmetric flows are removed from the

dynamics. The chance of the flow ‘visiting’ a TW in phase space is presumably related to how unstable it is (the spatial periodicity of the flow was also shortened to improve the stability of the TWs). In the notation of Duguet et al (2008a) the 3 new TWs were labelled as ‘A3’, ‘C3’ and ‘D2’ and all are mirror symmetric even though this symmetry was not imposed on the flow. A3 and C3 were readily converged in the continuation codes of Wedin & Kerswell (2004) but D2 could not be, due, we suspect, to insufficient axial resolution being achievable in the continuation code. The roll structure of D2 was, however, used to design a forcing in the homotopy approach which successfully yielded a similar-looking TW - called Z2 - with the same symmetries as D2. That they were, in fact, different waves became apparent when Z2 could not be continued below  $Re = 3250$  whereas D2 was discovered at  $Re = 2875$ . Despite the remarkably similar form of Z2 and D2 there was no readily apparent evidence that the calculated trajectories visited Z2 as well. The structure of A3, C3 and Z2 is shown in figure 1 compared to a known TW in S2, which, at  $\alpha = 1.25$  and  $Re = 2400$  is already known to be on the laminar-turbulent boundary (this is 2b\_1.25 in the nomenclature of Kerswell & Tutty 2007).

While all the new families possess the apparently universal features of exact coherent structures known in wall-bounded shear flows - wavy streaks with staggered quasistreamwise vortices (Nagata 1990, Waleffe 1998, 2001, 2003, Faisst & Eckhardt 2003, Wedin & Kerswell 2004, Pringle & Kerswell 2007), there are new structural features. A3 has a strikingly different cross-sectional profile from previously known TWs (e.g. S2) in that *both* fast and slow streaks are concentrated at the pipe wall leaving the interior relatively quiescent. Figure 1 shows that C3 and Z2 are also noticeably different too. However, it is the axial structure of Z2 which really makes it stand out. Z2 is not **S**-symmetric nor  **$\Omega$** -symmetric but *does* have mirror symmetry about two perpendicular planes. There is no *a priori* reason to expect the flow to prefer one of these symmetries over any of the others, but to date all TWs in *any* of the canonical shear flows have always been **S**-symmetric (except, trivially, the helical modes of PK07; Waleffe, private communication). Therefore Z2 and D2 are the first TWs found to possess only **Z**-symmetry (**Z** $_{\psi}$  with  $\psi$  suppressed as there is no longer an origin for  $\phi$ ). Both Z2 and D2 are, however, close to being **S**-symmetric in the sense that the simple indicator

$$\frac{\int_0^{2\pi} \int_0^1 (\mathbf{u} - \mathbf{S}\mathbf{u})^2|_{z=0} s \, ds \, d\phi}{\int_0^{2\pi} \int_0^1 (\mathbf{u} + \mathbf{S}\mathbf{u})^2|_{z=0} s \, ds \, d\phi} = O(10^{-3}). \quad (3.1)$$

In contrast, A3 and C3 are both **S**- and  **$\Omega_2$** -symmetric and hence also reflectionally symmetric about a plane at  $\pm\pi/4$  to the plane of shift-&-reflect symmetry (horizontal in figure 1) i.e. they have **Z** $_{\pm\pi/4}$  symmetry. The modes C3 and A3 appear through saddle node bifurcations at  $Re = 1141$  and  $1125$  with optimal wavenumbers  $\alpha^* = 1.2$  and  $2.0$  respectively giving these lowest saddle node bifurcations: see Table 1.

#### (a) Stability

An interesting feature of all TWs found so far is that they are unstable but only with an unstable manifold of very small dimension (invariably less than 10 for those checked so far - Faisst & Eckhardt 2003, Kerswell & Tutty 2007). The typical

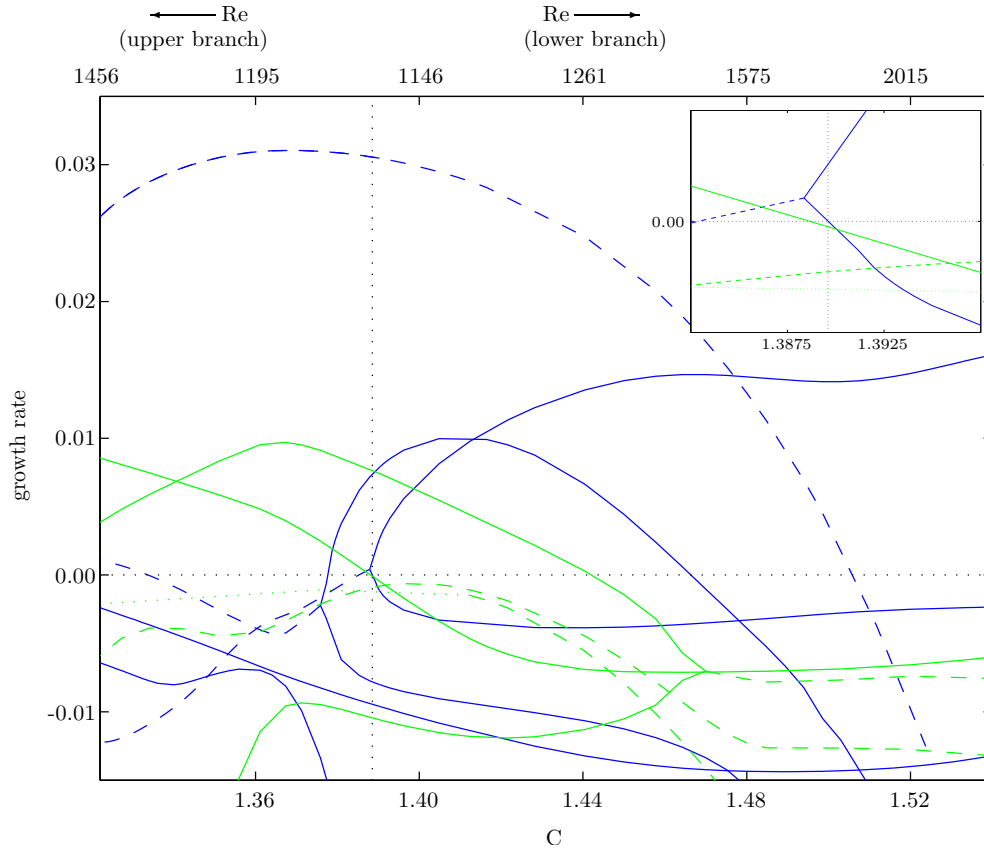


Figure 2. Stability of  $C3$  within the  $\mathbf{R}_2$ -symmetric subspace against the phase speed  $C$  as  $Re$  increases away from the saddle node bifurcation at 1141 ( $\alpha = \alpha^*$  and the corresponding  $Re$  is across the upper x-axis). Each line either indicates the locus of a real eigenvalue (solid) or a complex conjugate pair (dashed) as  $Re$  changes. The blue (dark) lines correspond to those which are symmetric under  $S$  while the green (light) lines are anti-symmetric under  $S$ . The vertical dotted line indicates the saddle node, of which the inset is a close up. There are bifurcations at  $C = 1.33, 1.38, 1.385, 1.388, 1.389$  (the saddle node),  $1.44, 1.47$  and  $1.50$ . (The dotted green indicates a stable complex conjugate pair which was difficult to resolve).

situation is illustrated in figure 2 which displays the spectrum of the  $C3$  wave at the wavenumber  $\alpha^*$  which gives its lowest saddle node bifurcation. While  $Re$  is double-valued near the bifurcation, the phase speed  $C$  is not, monotonically decreasing in value as the bifurcation point is crossed from lower to upper branch. This then provides a convenient abscissa to show how the stability changes on both upper and lower branches as  $Re$  increases away from the saddle node bifurcation. The stability analysis is restricted to disturbances periodic over the same wavelength as  $C3$  and to the  $\mathbf{R}_2$ -symmetric subspace.  $C3$  is particularly interesting for two reasons: 1) the number of unstable directions decreases down to one for  $Re \geq 1826$  along the lower branch and 2) there are several bifurcations involving real eigenvalues. The implication of the first observation is that  $C3$ , which is on the laminar-turbulent boundary

(in a pipe 2.5D long at  $Re = 2400$ , Duguet et al 2008a), will become a local attractor there beyond  $Re = 1826$  for  $\mathbf{R}_2$ -symmetric flow (confirmed by Duguet et al. 2008a). The second observation means that new solutions bifurcate from C3 which are also steady in an appropriately translating Galilean frame - that is to say they are travelling waves (this is how the asymmetric travelling wave of Pringle & Kerswell 2007 appears). There are 4 possibilities: a transcritical bifurcation where no symmetry is broken, and 3 types of symmetry-breaking pitchfork in which only  $\mathbf{S}$ -,  $\mathbf{Z}$ - or  $\mathbf{\Omega}_2$ -symmetry is retained (recall 2 symmetries imply the third). For example, the bifurcation at  $(C, Re) = (1.47, 1449)$  is a mirror-symmetry-breaking pitchfork which can be followed to produce TWs which resemble those of S2. D2 is plausibly the product of a  $\mathbf{S}$ -symmetry-breaking pitchfork from C3 (D2 is very similar to Z2 and hence C3 - see figure 1).

Hopf bifurcations are the generic scenario, however, leading to more complicated relative periodic orbits such as that traced by Duguet *et al.* (2008b). Within the  $Re$  range of figure 2, C3 experiences 3 Hopf bifurcations (corresponding to the phase speeds  $C = 1.33, 1.385, 1.50$ ) which all lead to relative periodic orbits with the  $\mathbf{Z}$ -symmetry broken.

#### 4. New mirror-symmetric classes

The fact that A3 and C3 appear at such low  $Re$  strongly suggests that there are analogous waves in different rotational symmetry classes also existent at pre- and transitional  $Re \lesssim 2400$ . This indeed turns out to be the case with all except two (N1 and M3) of the new families of mirror-symmetric modes being easily found using the homotopy approach once the appropriate  $\mathbf{\Omega}$ -symmetry is incorporated. Selecting a roll forcing of  $\mathbf{R}_{2m}$ -symmetry defined, in the notation of Wedin & Kerswell (2004), by  $\lambda_{2m} i$  with  $i = 2, 3$  or 4 invariably led to a subharmonic  $\mathbf{R}_m$ -symmetric streak instability which could be easily tracked back to zero forcing (this was exactly the strategy used by Wedin & Kerswell to find their  $\mathbf{R}_1$  TW ('U1' in Table 1) which has two roll pairs and appears beyond  $Re = 3046$ ). The remaining waves were found by homotopy in  $m$  starting from a M5 wave (M5 being used rather than M4 because of the similar parity of velocity components with respect to the radius) for M3 and from N3 for N1.

Inspection of the new lower branch TWs - see Figures 3 and 4† - indicates that the mirror-symmetric family reported in Pringle & Kerswell (2007) is the first family (M1) and A3 is a member of the second family (M2) of a class of TWs ( $Mm$  with  $m=2,3,4,5,6, \dots$ ) with two layers of fast and slow streaks. Furthermore, C3 appears a member of a  $\mathbf{R}_2$ -symmetric family (N2) of another class of TWs ( $Nm$  with  $m=1,2,3,4,5, \dots$ ). In Figure 5, M1 and N1 are shown alongside U1 found in Wedin & Kerswell (2004). That all three travelling waves can possess exactly the same symmetries yet have such different velocity fields and locations in phase space is in itself noteworthy. The ordering of the families within the respective classes is clear from the phase speed and wavenumber data shown in figure 6.

Extending the  $Mm$  and  $Nm$  families to higher rotational symmetry  $m$  is straightforward and could have been continued if the general trend had not already emerged. The bifurcation points for  $Mm$  TWs monotonically increase in  $Re$  and decrease in

† Videos to be included with this paper are particularly illuminating - they are currently on <http://www.maths.bris.ac.uk/~cp1571/TWs/table.htm>



Source		Symmetries	$Re_{lowest}$	$\alpha^*$
FE03 & WK04	S2	$\mathbf{S} \ \& \ \mathbf{R}_2$	1358	1.55
	S3	$\mathbf{S} \ \& \ \mathbf{R}_3$	1251	2.44
	S4	$\mathbf{S} \ \& \ \mathbf{R}_4$	1647	3.23
	S5	$\mathbf{S} \ \& \ \mathbf{R}_5$	2485	4.11
	WK04	S6	$\mathbf{S} \ \& \ \mathbf{R}_6$	2869
	U1	$\mathbf{S} \ \& \ \mathbf{\Omega}_1$	3046	2.17
PK07	Asymm (S1)	$\mathbf{S}$	$\approx 820$	$\approx 1.8$
	Mirror-symm (M1)	$\mathbf{S} \ \& \ \mathbf{\Omega}_1$	773	1.44
	Helical	$\mathbf{\Omega}_1^\beta$	773	1.44
here	M2 (A3)	$\mathbf{S}, \ \mathbf{R}_2 \ \& \ \mathbf{\Omega}_2$	1125	2.0
	M3	$\mathbf{S}, \ \mathbf{R}_3 \ \& \ \mathbf{\Omega}_3$	1552	2.2
	M4	$\mathbf{S}, \ \mathbf{R}_4 \ \& \ \mathbf{\Omega}_4$	1824	2.6
	M5	$\mathbf{S}, \ \mathbf{R}_5 \ \& \ \mathbf{\Omega}_5$	2143	3.1
	M6	$\mathbf{S}, \ \mathbf{R}_6 \ \& \ \mathbf{\Omega}_6$	2531	3.5
	N1	$\mathbf{S} \ \& \ \mathbf{\Omega}_1$	1554	0.9
	N2 (C3)	$\mathbf{S}, \ \mathbf{R}_2 \ \& \ \mathbf{\Omega}_2$	1141	1.2
	N3	$\mathbf{S}, \ \mathbf{R}_3 \ \& \ \mathbf{\Omega}_3$	1037	2.0
	N4	$\mathbf{S}, \ \mathbf{R}_4 \ \& \ \mathbf{\Omega}_4$	1290	2.5
	N5	$\mathbf{S}, \ \mathbf{R}_5 \ \& \ \mathbf{\Omega}_5$	1622	2.9
	Z2	$\mathbf{Z} \ \& \ \mathbf{R}_2$	$\approx 3250$	0.8
	D2	$\mathbf{Z} \ \& \ \mathbf{R}_2$	$< 2875$	?

Table 1. The various symmetries of all the TWs currently known. FE04 is Faisst & Eckhardt (2003), WK04 is Wedin & Kerswell (2004) and PK07 is Pringle & Kerswell (2007). The ‘U1’ entry for WK04 is to indicate that they found a mirror-symmetric TW with no rotational symmetry but at high  $Re$  and its relationship to M1 and N1 is unclear.

$C$  with  $m$  (see figure 7). This behaviour is not strictly true for the N-class where N3 has the lowest bifurcation point but is for  $m > 3$ . The fact that N3 appears before N1 and N2 mimics the situation for the original S-class (non-mirror symmetric) TWs. The fact that the lowest  $Re$  for each  $Nm$  family as well as M1 and M2 is consistently smaller than the lowest  $Re$  for the equivalent  $Sm$  family is suggestive that the latter all bifurcate off the former in mirror-symmetry-breaking bifurcations. So far, there are two known examples of this: S1 (the asymmetric waves of Pringle & Kerswell 2007) bifurcating off M1, and a recently-discovered connection between S3 and N3 at  $Re = 2400$ . This latter bifurcation is shown clearly in a  $\Lambda$  vs  $\alpha$  plot (see Figure 8) which also highlights the smoothness of the N3 curve compared to the convoluted nature of the S3 curve (first plotted in Figure 1 of Kerswell & Tutty

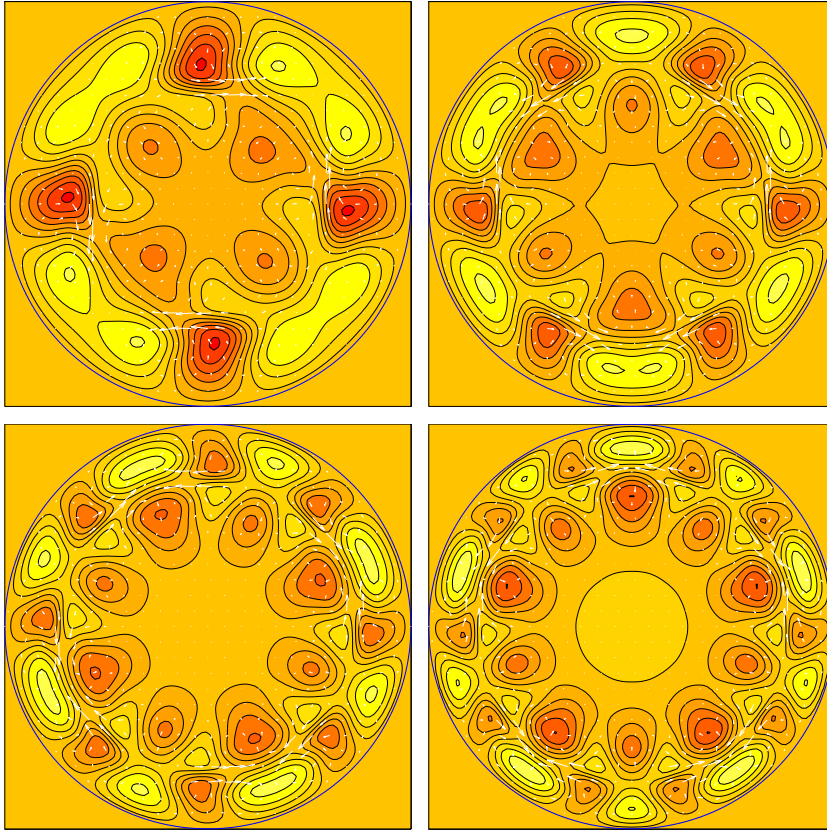


Figure 3. Slices of the lower branch M-class TW solutions at  $Re = 2400$  and at each travelling wave's respective value of  $\alpha^*$ . M2 (A3, top left), M3 (top right), M4 (bottom left) and M5 (bottom right). Contour levels of the streamwise velocity perturbation are in increments of  $0.02U$  running from  $-0.19U$  (dark red) to  $0.19U$  (white).

2007). Another important feature is the very large friction factors (compared to the typical turbulent values) achieved by the upper branch N3 TWs. This is a general property of the N-class as shown in Figure 9. The new upper branch N- and M-class TWs - see figure 10 - all show an expected intensification of the slow and fast streaks, and the positioning of the fast streaks closer to the pipe wall than the corresponding lower branch solutions. This localisation is particularly noticeable for the N-class TWs and leads to their large friction factors. Figure 9 also shows that the friction factors associated with some of the lower branch N-class TWs as well as M1 and M2 are significantly higher than time-averaged experimental values (as shown by the upper dotted line) and those of the S-class TWs.

Finally, it's worth remarking that the M-class TWs start to appear at fascinatingly low  $Re$  given their intricate double radial layer structure: for example, M4 appears at  $Re = 1824$  (see figure 7 and Table 1). That they exist at all is not a surprise (presumably TWs with three radial layers are possible too), but that they appear so early in  $Re$  surely is and contrasts starkly with Z2 which doesn't emerge until  $Re \approx 3250$ .

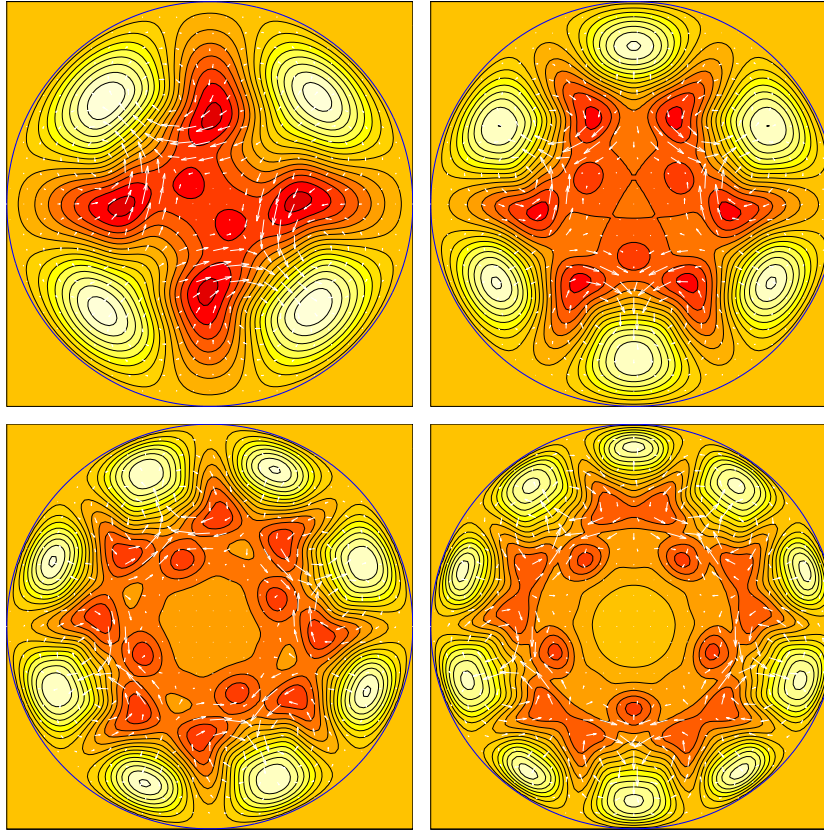


Figure 4. Slices of the lower branch N-class TW solutions at  $Re = 2400$  and at each travelling wave's respective value of  $\alpha^*$ . N2 (C3, top left), N3 (top right), N4 (bottom left) and N5 (bottom right). Contour levels of the streamwise velocity perturbation are in increments of  $0.02U$  running from  $-0.19U$  (dark red) to  $0.19U$  (white).

## 5. Discussion

In this paper we have described 2 new classes of mirror-symmetric TWs - the M- and N-classes which are also both shift-&-reflect symmetric - and a new mirror-symmetric family Z2 which is not. The M- and N-class waves appear earlier in  $Re$  than the original non-mirror-symmetric waves of Faisst & Eckhart (2003) and Wedin & Kerswell (2004) suggesting that the latter are borne through generally supercritical symmetry-breaking bifurcations from them. This was already found to be the case for the M1 waves (Pringle & Kerswell 2007) but now seems more generally true now that the various TW families making up each class have been found. The stability analysis of N2 presented here provides a timely reminder of the bifurcation possibilities: **S** or **Z** symmetries can be broken at pitchfork bifurcations leading to either **Z**-class TWs or **S**-class TWs found originally (Faisst & Eckhardt 2003, Wedin & Kerswell 2004). Hopf bifurcations, of course, give rise to periodic orbits in the Galilean frame of the TW or quasiperiodic orbits in the pipe (laboratory) frame. Tracing these requires a more sophisticated approach based upon

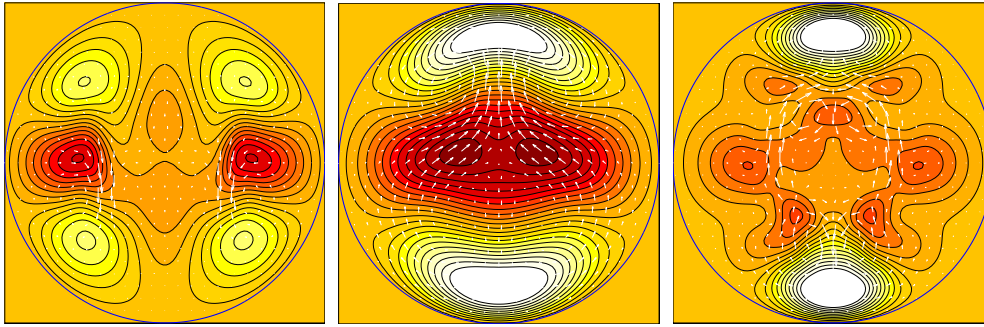


Figure 5. Slices of the lower-branch, non-rotationally-symmetric TW solutions M1 (left), N1 (middle) and U1 (right). The first two are at  $Re = 2400$  while the last is at  $Re = 3046$ , and each travelling wave is shown at its respective value of  $\alpha^*$ . Contour levels of the streamwise velocity perturbation are in increments of  $0.02U$  running from  $-0.19U$  (dark red) to  $0.19U$  (white).

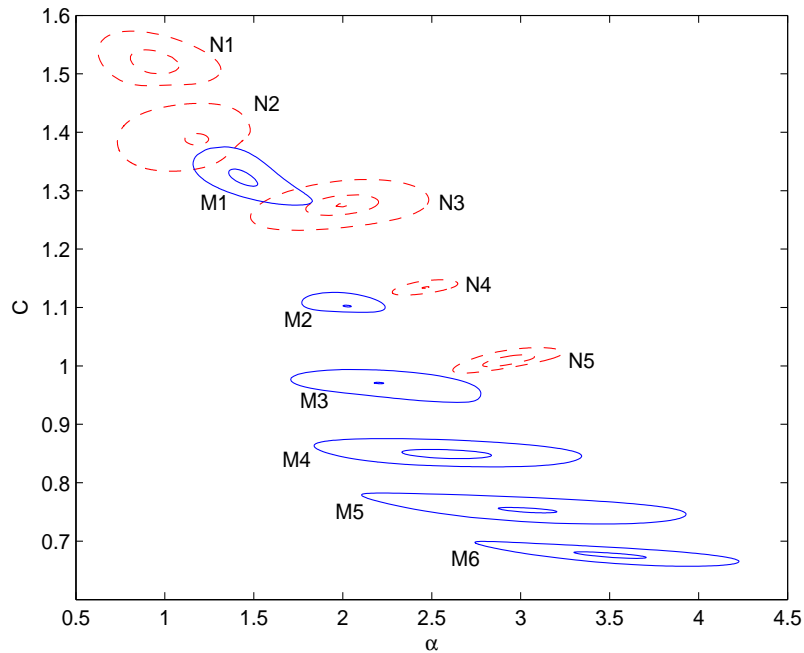


Figure 6. A plot of the phase speed  $C$  (in units of  $U$ ) against the axial wavenumber  $\alpha$  (in units of  $2/D$ ). The contours correspond to different  $Re$  with the loops (M-class/N-class TWs shown using blue solid/red dashed lines) constricting as they move towards the saddle nose indicating  $\alpha^*$ . The various  $Re$  for each TW are as follows: M1 776 & 820, M2 1125 & 1145, M3 1552 & 1650, M4 1846 & 2037, M5 2148 & 2400, M6 2540 & 2662; N1 1581 & 1747, N2 1150 & 1318, N3 1038, 1050 & 1120, N4 1292 & 1300, N5 1629 & 1652.

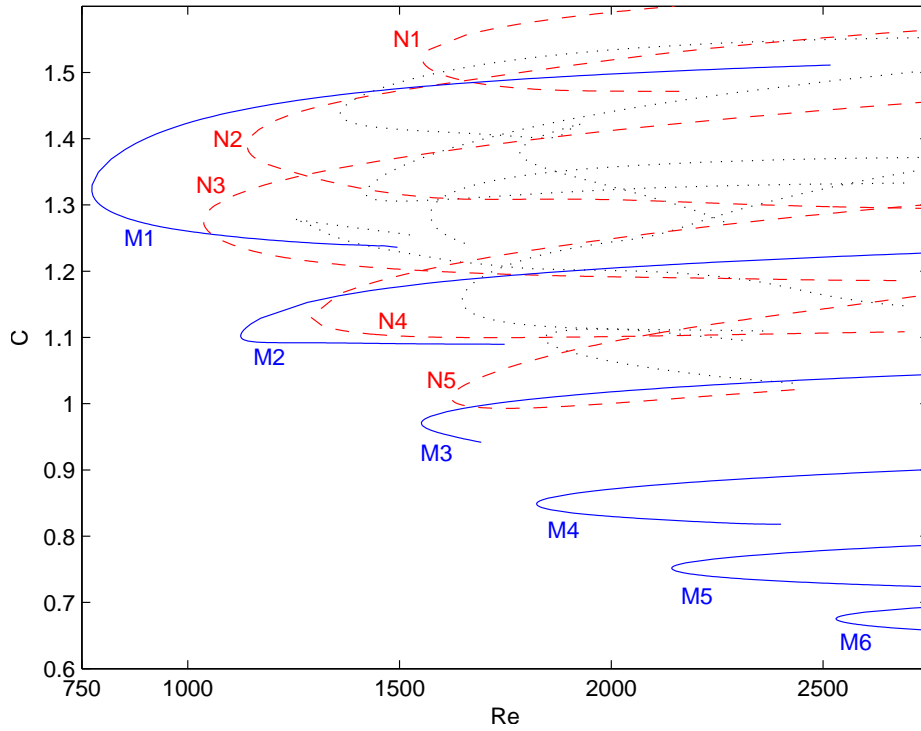


Figure 7. A plot of  $C$  against  $Re$  for  $\alpha^*$  (see Table 1) which gives the lowest saddle node bifurcation for each TW (M-class/N-class waves shown using blue solid/red dashed lines). The non-mirror symmetric S-class of Faisst & Eckhardt(2003) and Wedin & Kerswell (2004) which generally appear at higher  $Re$  are shown using black dotted lines. Typical resolutions used (in the nomenclature of Wedin & Kerswell (2004)) are: M1 (14,25,5), M2 (12,30,6), M3 (10,30,10), M4 (10,30,8), M5 & M6 (10,35,8); N1 (14,30,5), N2 (12,30,6), N3 & N4 (10,30,8), N5 (8,35,8).

time-stepping the equations and searching for an exact return of an appropriately chosen Poincaré map (Viswanath 2007, 2008, Duguet et al 2008b).

The original motivation for searching for all the new TWs discussed here was the discovery of previously-unknown TWs in the laminar-turbulent boundary by Duguet et al (2008a). It is likely that some of the lower branch solutions of these new waves similarly populate this boundary so that their stable manifolds also play a role in determining if a given initial condition will lead to relaminarisation or a turbulent evolution. There are, however, many interesting issues surrounding this assumption. For example, if a lower branch TW is on the boundary at one  $Re$ , will it still be at  $10Re$  and if not, how did it leave? In a long pipe where the TWs are continuously parametrised by their wavenumber over a finite range, at what critical wavenumber does the lower branch TW leave the boundary on its way to becoming an upper branch TW? The asymmetric wave (Pringle & Kerswell 2007) provides an obvious example being on the boundary for  $\alpha = 0.625$  (at  $Re = 2875$ , Schneider et al 2007b, Duguet et al 2008a) but presumably not for  $\alpha = 1.44$  where its wall shear stress is high (see figure 6 of Pringle & Kerswell 2007). Hopefully these issues will be discussed elsewhere in this celebratory volume.

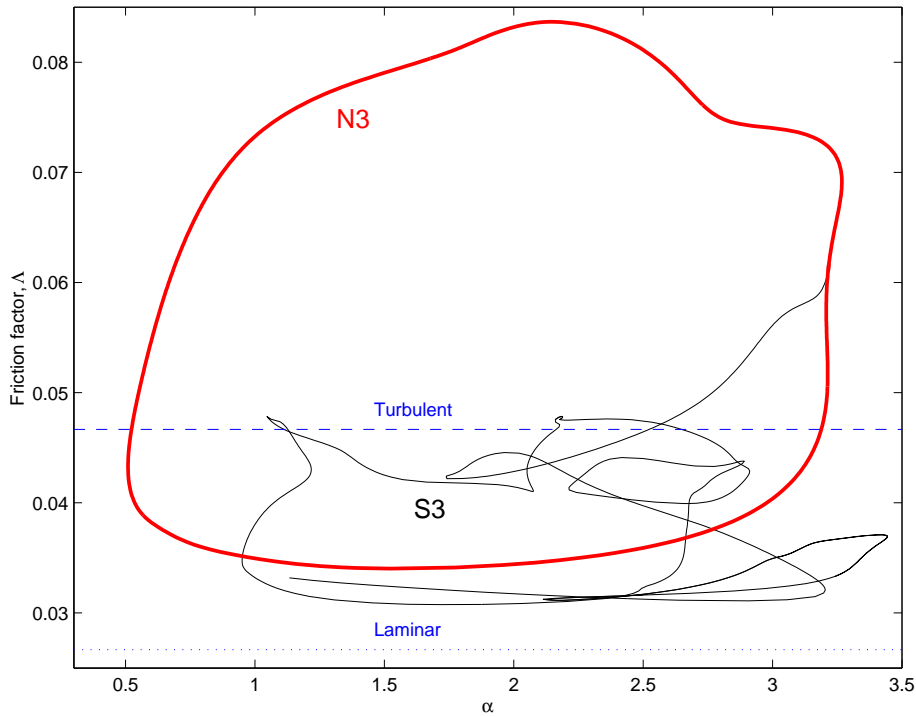


Figure 8. The  $S3$ - $N3$  family connection at  $Re = 2400$  on a friction factor  $\Delta$  vs  $\alpha$  plot. This plot for  $S3$  appeared in Figure 1 of Kerswell & Tutty (2007). Now the bifurcation from the more symmetric  $N3$  family is clear. The resolution is (8,30,6) in the nomenclature of Wedin & Kerswell (2004) throughout (the  $N3$  curve is essentially unchanged for the higher resolution of (10, 30, 8)). The lower dotted line represents the lower bound given by the Hagen-Poiseuille solution ( $\Delta_{lam} = 64/Re$ ) and the upper dashed line corresponds to the  $Re = 2400$  value of the log-law parameterization of the experimental data  $1/\sqrt{\Delta} = 2.0 \log(Re\sqrt{\Delta}) - 0.8$  (see Schlichting 1968, equation (20.30)).

The discovery that the upper branch solutions of the new N-class waves have such high wall shear stresses is, however, potentially more important. A recent attempt to extract coherent fast-streak structures from pipe turbulence (Willis & Kerswell 2008) concluded that the TWs currently known were not energetic enough to be part of the turbulent attractor as previously supposed and speculated about others as-yet-undiscovered. The new N-class of mirror-symmetric TWs may well be these missing waves as illustrated so starkly by Figure 8 and the issue clearly needs to be revisited.

In this report, we hope a step forward has been made in appreciating the hierarchy of TWs which exist in pipe flow. Generally the picture is that TWs with shift-&-reflect symmetry, mirror-symmetry and a low degree of rotational symmetry seem to appear first and spawn further, less symmetric, TWs through bifurcations as  $Re$  increases. However, given the notorious complexity of the Navier-Stokes equations, it would be foolhardy to be too rigid about this. What should be absolutely clear, though, is that the pipe flow problem continues to fascinate and intrigue us fully 125 years after Reynolds' original experiments.

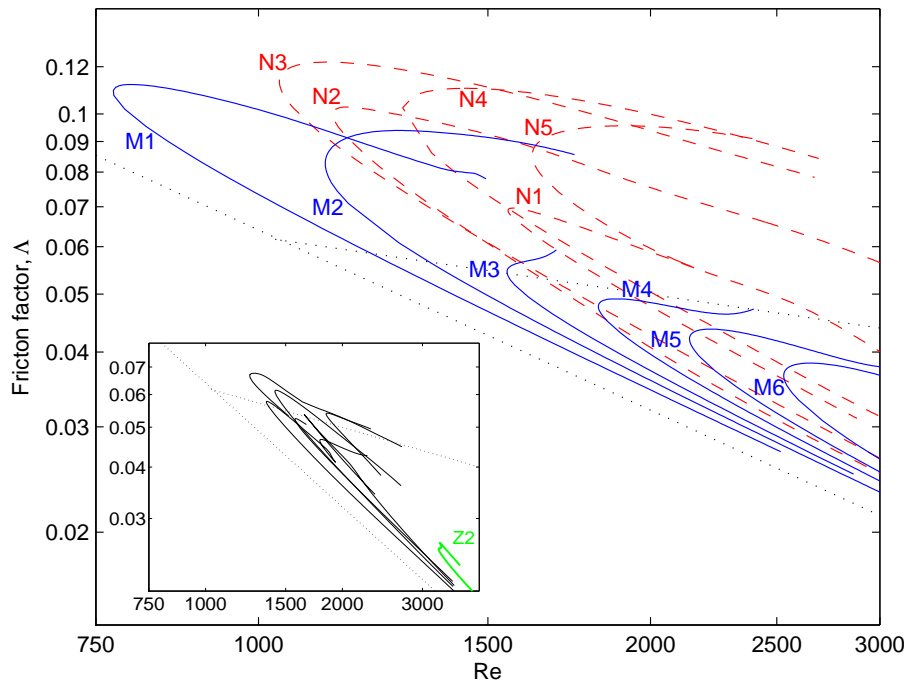


Figure 9. The friction factor,  $\Lambda$ , against  $Re$  for the newly found travelling waves, as well as the asymmetric wave (M1). The lower dotted line corresponds to the laminar state, and the upper dotted line to the log-law parameterisation of experimental data,  $1/\sqrt{\Lambda} = 2.0 \log(Re_m \sqrt{\Lambda}) - 0.8$  (see Schlichting 1968, equation (20.30)). The inset shows the same plot but for the travelling waves S2–S6 (only selected curves drawn for clarity) and Z2 which appears at much higher  $Re$ . The earlier onset in  $Re$ , and significantly higher friction factors of the new  $M$  and  $N$ -class travelling waves is clearly apparent.

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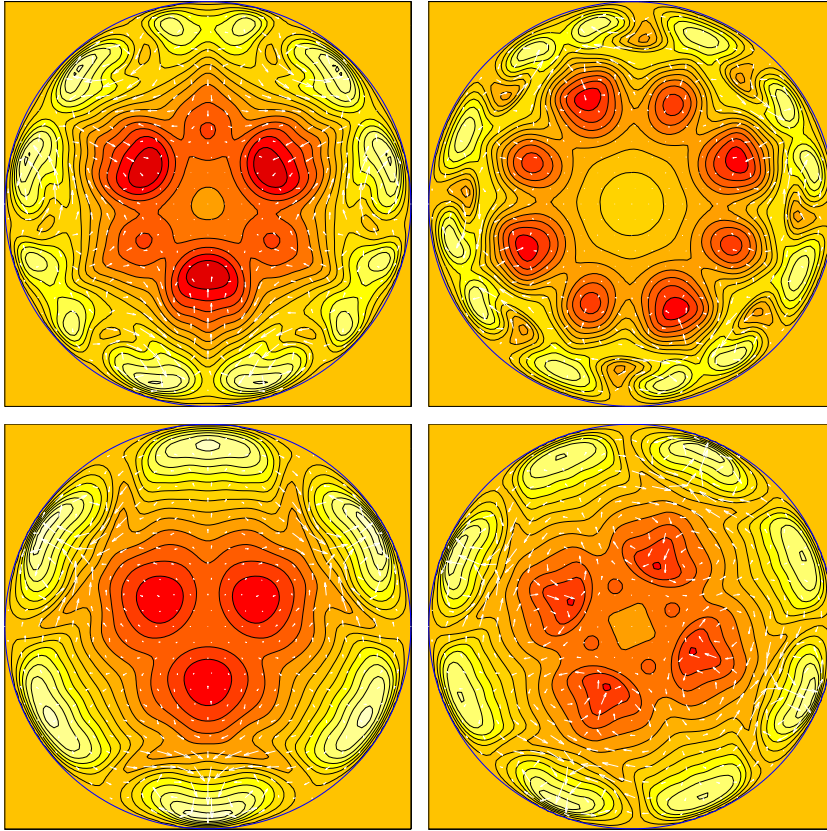


Figure 10. Slices of upper branch solutions at  $Re = 2400$  and at each travelling wave's respective value of  $\alpha^*$ . M3 (top left), M4 (top right), N3 (bottom left) and N4 (bottom right). Contour levels as in figures 3 and 4.

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