Lecture 8: Edge Tracking – Walking the Tightrope

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In the subcritical transition scenario, there is typically a regime where the linearly stable basic state coexists with the turbulent state. The simplest model of this is a 1D system $\dot{y} = y(\frac{1}{2} - y)(y-1)$ where the basic state is $y=0$, the 'turbulent' state is $y=1$ and there is an intermediate state $y = \frac{1}{2}$ which separates initial conditions which become turbulent (if $y(0) > \frac{1}{2}$ then $y(t) \to 1$ as $t \to \infty$) from those which laminarise (if $y(0) < \frac{1}{2}$ then $y \to 0$ as $t \to \infty$). The state $y = \frac{1}{2}$ is special in that it separates these two different types of behaviour and corresponds to neither (if $y(0) = 1/2$, $y(t) = 1/2\ \forall\, t$). Identifying initial conditions which lead to this intermediate behaviour is crucial for understanding transition. An extra dimension is needed in our model to allow this intermediate behaviour to be non-trivial so consider the 2D system

$$\begin{aligned} \dot{x} &= -x + 10y, \\ \dot{y} &= y(10\mathrm{e}^{-0.01x^2} - y)\,(y-1)\,, \end{aligned} \tag{1}$$

which, like the 1D model, has two stable equilibria and an unstable state. The line $y=1$ separates initial conditions which will laminarise to $(0,0)$ and those which become 'turbulent', that is, spiral into the fixed point at $\approx (14, 1.4)$: see figure 1. All points on $y=1$ – here the basin boundary for the laminar and turbulent states or more generally called the 'edge' – are attracted to the relative attractor at $(10,1)$ which is called the 'edge state' (this is a saddle point in 2D but an attractor for trajectories confined to the 1D edge). This lecture is about recent successes in identifying the edge state in realistic flows whereas the next will be concerned with identifying the closest point of approach of the edge to the laminar state[1].

1 Approach

The approach to finding the edge goes back to [17] (at least in fluid mechanics) and is based on simple bisection combined with direct numerical simulation (DNS). A turbulent velocity field $\mathbf{u}_{turb}(\mathbf{x})$ generated by DNS is the starting point and is decomposed into a 2D part and a 3D part. Importantly, this 2D part is known in practice not to be able to initiate 3D turbulent state in plane Couette or pipe flow. In pipe flow, either azimuthally averaging or

[1] This will indicate the initial conditions of least energy which can trigger turbulence - if this distance is defined as $\sqrt{x^2+y^2}$ and energy as x^2+y^2, this would be the point $(0,1)$ here.

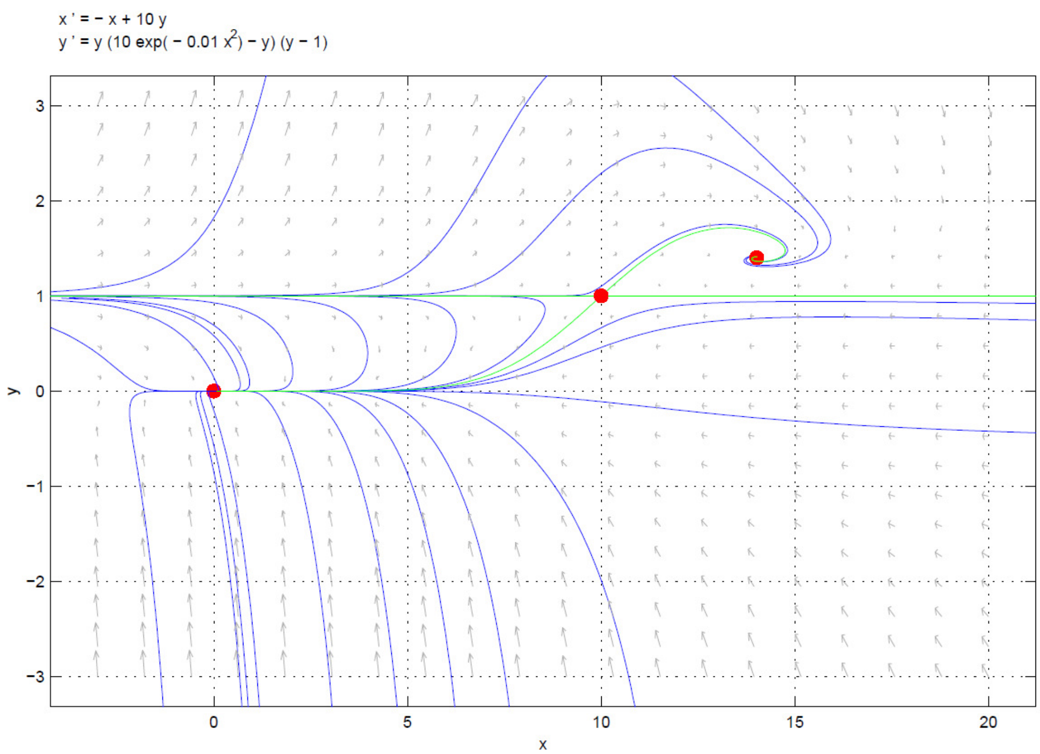

Figure 1: Phase plane of the 2D system (1) created using the free Matlab macro pplane6.m written by John Polking, Rice University, 1995. The laminar state is at $(0, 0)$, the edge state at $(10, 1)$ and the 'turbulent' state at $\approx (14, 1.4)$. The edge is the line $y = 1$.

axial averaging can be used, i.e.

$$u_{2D} = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta,z)\mathrm{d}\theta \\ \frac{1}{L}\int_0^{L} u(r,\theta,z)\mathrm{d}z \end{cases} \tag{2}$$

and then

$$\mathbf{u}_{3D}(\mathbf{x}) := \mathbf{u}_{turb}(\mathbf{x}) - \mathbf{u}_{2D}. \tag{3}$$

The key point is that these two velocity fields can be used to reconstruct an initial condition as follows

$$\mathbf{u}(\mathbf{x},0;\lambda) := \mathbf{u}_{2D} + \lambda\mathbf{u}_{3D}(\mathbf{x}) \tag{4}$$

where $\lambda \in [0,1]$ is a free mixing ratio. $\mathbf{u}(\mathbf{x},0;1) = \mathbf{u}_{turb}$ so that the DNS code will continue to follow the turbulent attractor whereas $\mathbf{u}(\mathbf{x},0;0) = \mathbf{u}_{2D}$ and the flow must subsequently relaminarise. Given these contrasting behaviours, there must be (at least) one intermediate value λ^* where the flow neither becomes turbulent nor relaminarises. At this value $\mathbf{u}(\mathbf{x},0;\lambda^*)$ must be precisely on the edge and the flow has to subsequently walk the tightrope of the edge neither converging back to the laminar state nor swinging up to the turbulent state. The approach to finding λ^* is to successively bisect the n^{th} interval $[\lambda_l^n, \lambda_u^n]$ where λ_l causes relaminarisation and λ_u leads to turbulence. This requires initialising a DNS using $\mathbf{u}(\mathbf{x},0;\frac{1}{2}(\lambda_l+\lambda_u))$ and having some criterion for deciding if the flow is starting to relaminarise or become turbulent (obviously the quicker this decision is made, the faster the bisection technique). Typically, λ^* must be known to *considerable* accuracy to be able to track the edge for any length of time because of the edge's inherent instability. To illustrate this, let μ be the Lyapunov exponent which describes the rate at which trajectories diverge from the edge. Then if $|\lambda_u^n - \lambda_l^n| \le 10^{-14}$ (e.g. double precision accuracy in Fortran), one could anticipate being able to shadow the edge for a time T where

$$e^{\mu T} \times 10^{-14} \lesssim 1. \tag{5}$$

Typically, for pipe flow at $Re = O(2500)$, $UT/D \approx 200-300$ (D pipe diameter and U bulk velocity along the pipe). Once λ^* has been found to this accuracy, the bisection approach can be restarted a little back from T, say $0.9T$, using the velocity fields $\mathbf{u}(\mathbf{x},0.9T;\lambda_l)$ and $\mathbf{u}(\mathbf{x},0.9T;\lambda_u)$ so that both velocity fields are still very similar albeit on different 'sides' of the edge. In this way a long-time edge-shadowing trajectory can be generated by piecing together different tracking sections. The edge-tracking technique relies on two ingredients for success.

1. The ability to quickly distinguish between two (and only two) different evolutions. A clear energy level separation between the turbulent state and the edge is usually sufficient.

2. The fact that the edge is a hypersurface and so can be reached by adjusting one parameter (or alternatively, there is only one unstable direction across the edge).

It's worth remarking here that this bisection approach can identify more general hypersurfaces in phase space than basin boundaries since only different *initial* behaviours are needed. Whether the turbulent state is sustained or not is irrelevant for the mechanics of the procedure although this does, of course, influence the interpretation of the edge.

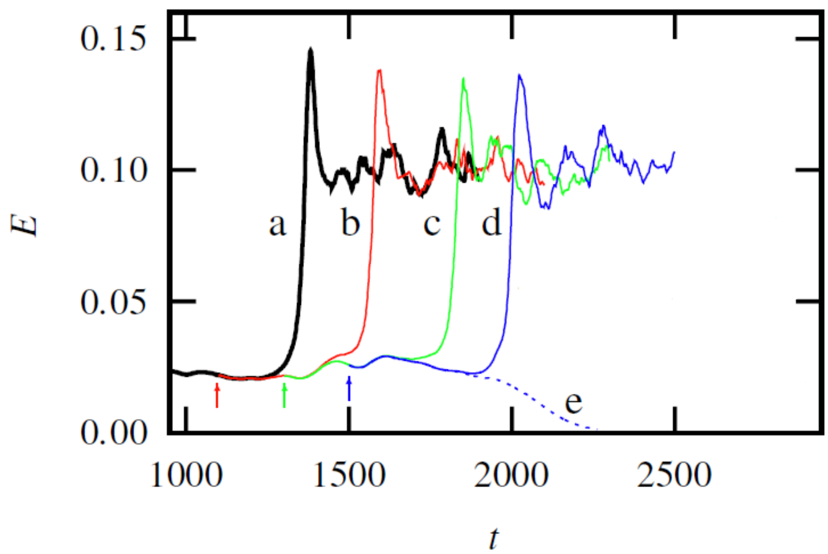

Figure 2: Perturbation energy traces of trajectories bounding the edge of chaos from [12]. Re=2875 and the pipe is $5\,D$ long.

2 First Calculations

Following Toh & Itano's [17] first calculations, Skufca, Yorke & Eckhardt [16] treated a low-dimensional model of shear flow and then the first edge-tracking calculation in pipe flow was carried out in [12]. Figure 2 shows one of their edge trackings in a $5\,D$ pipe at $Re = 2875$. The first thing to notice is the characteristic way the trajectories leave the edge to swing up in energy to the turbulent state. The trajectory (d) and the relaminarising trajectory (e) both provide a good approximation to the edge dynamics until about $t \approx 1800$ when they start to separate. If the edge is followed for longer, chaotic dynamics are obtained as the 'edge state' (the limiting set). Interestingly, a time-average of the velocity field (see figure 3) over this attractor reveals a coherent large scale structure where streamwise rolls and streaks combine in a familiar way [4, 20]. A very similar-looking travelling wave (TW) was found soon afterwards [9] prompting the conclusion that this TW must be embedded in the edge state. The fact that some TWs could be part of the edge was an idea already gaining momentum [5, 19]. Figure 4 shows the effect of perturbing a typical lower branch TW along its most unstable direction (from [5], 'lower branch' meaning closer to the laminar state than the 'upper branch' in any sensible measure such as energy or dissipation). Depending on the sign of the perturbation, the evolution can either lead to the turbulent state (solid green line) or the laminar state (dashed green line). Hence this TW must be on the edge. Contrast this situation with perturbing an upper branch TW (figure 5) where the subsequent flow is turbulent whichever way the perturbation is applied. The consensus now is that the edge is made up of (at least) the union of all the stable manifolds of the lower branch TWs.

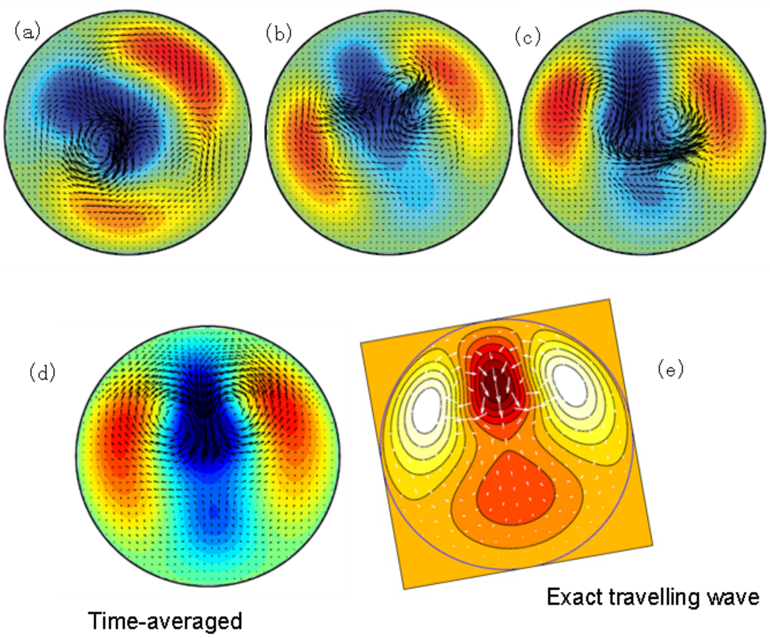

Figure 3: Edge trajectory snapshots, their time-average and an exact travelling wave calculated in [9].

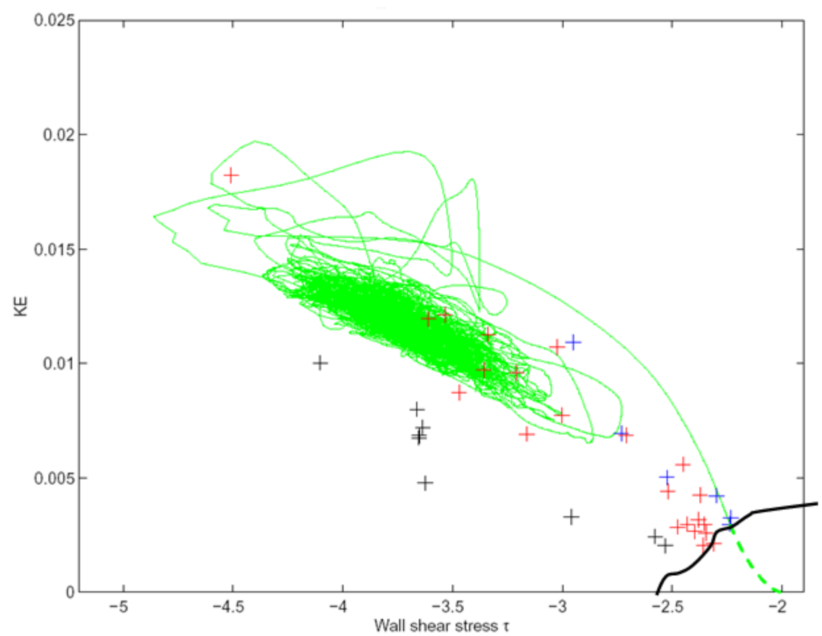

Figure 4: The disturbed kinetic energy per unit mass versus wall shear stress τ for a lower branch TW. The subsequent flow either becomes turbulent (solid green line) or tamely relaminarises (dashed green line). All the TWs known in 2007 to fit into a $5\,D$ pipe are plotted: 2-fold rotationally symmetric TWs(blue), 3-fold TWs(red) and 4-fold TWs(black). The non-smooth bold black separating line is supposed to indicate the edge. For details see [5].

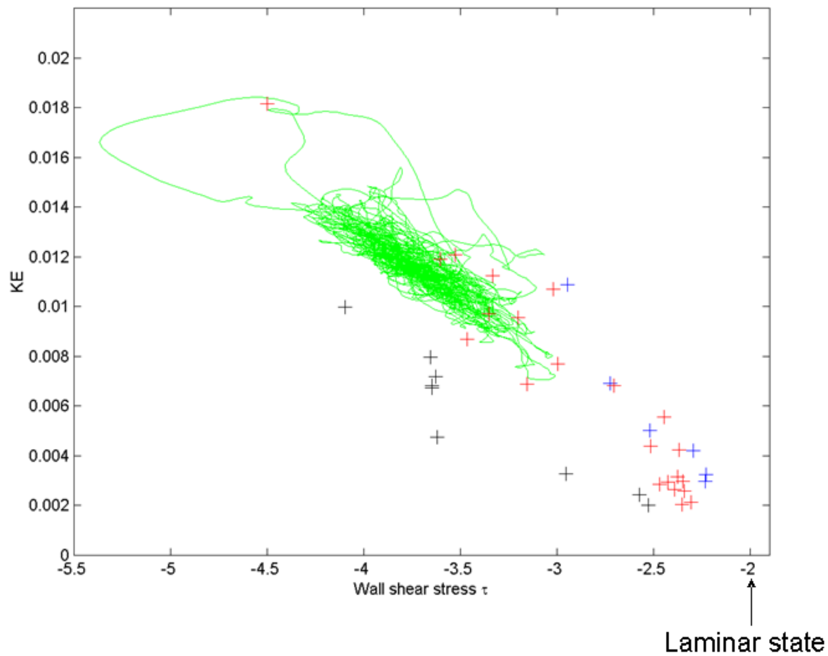

Figure 5: The disturbed kinetic energy per unit mass versus wall shear stress τ for a typical upper branch TW [5].

3 Developments

The significance of the edge state is that initial conditions which are just able to trigger transition should follow the edge, reach the edge state and then be ejected along its unstable manifold to the turbulent state. Hence, in some sense, it is an organising centre for transition (see [3] for a discussion). From another perspective, the edge-tracking technique can be viewed as a powerful new approach to finding nonlinear solutions of the Navier-Stokes equations. The edge state can change depending on the flow geometry (e.g. length of pipe), Re and the symmetries imposed on the flow (see examples below). Starting conditions for the tracking will also be important if the edge state is not the unique attractor on the edge. All of these aspects of the procedure have been explored and will now be briefly described.

3.1 Varying Re

Schneider & Eckhardt [11] examined how the typical edge state energy varies compared to the turbulent state for $Re \in [2000, 4000]$ in their $5\,D$ pipe. Figure 6 makes it clear that the separation of the edge and turbulence increases with Re. However, over this range of Re, the edge state remains chaotic (see [3] for even higher Re).

3.2 Geometry and use Invariant Subspaces

Duguet, Willis & Kerswell [2] looked at the edge state in a $2.5\,D$ pipe imposing R_2 symmetry (the flow is symmetric under a π rotation about its axis) at $Re = 2400$. The plot of E_{3d}

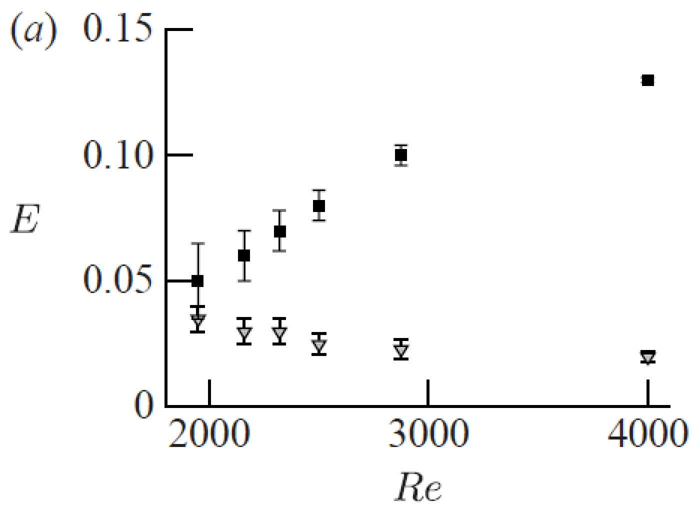

Figure 6: Typical energy of the edge trajectory (down triangles) and turbulence (black boxes) as a function of Re for a $5\,D$ pipe from [11].

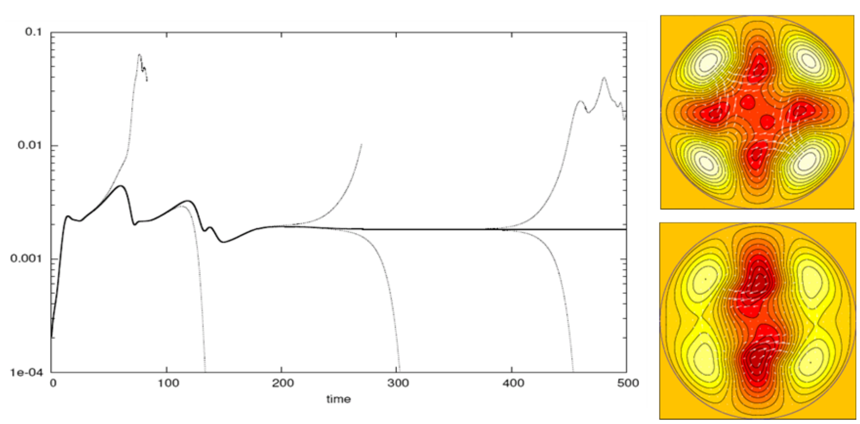

Figure 7: Energy contained in the axially dependent flow. The thick line indicates the edge trajectory and the thinner lines nearby trajectories which either relaminarize (energy decrease) or become turbulent (energy increases to a higher level). Pipe length is $2.5\,D$, $Re = 2400$ and the flow is calculated within the R_2-subspace [2]. The two cross-sections on the right indicate (at least) two travelling waves which are stable on the edge (the upper one corresponds to the trace on the left at large times).

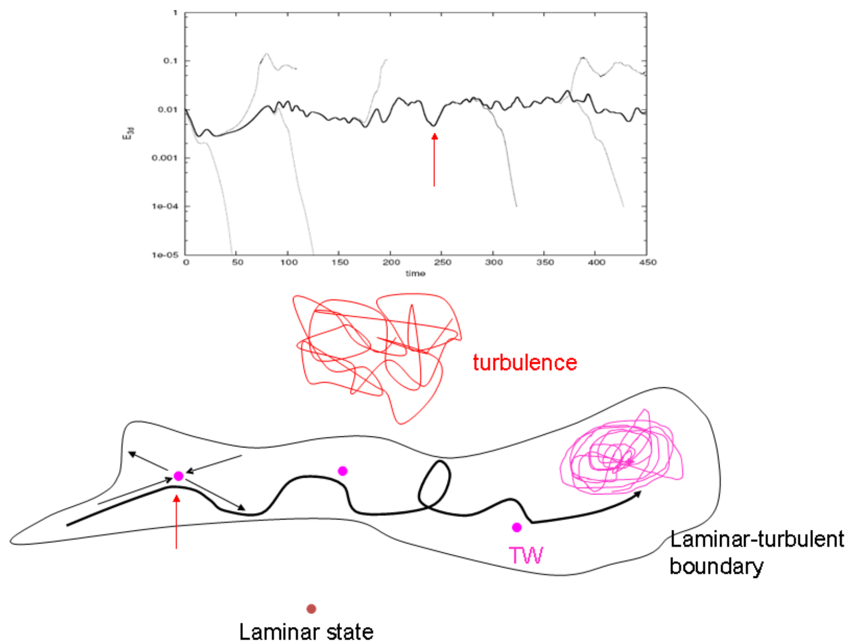

Figure 8: **Upper:** energy contained in the axially dependent modes on the edge for 5D pipe at $R_e = 2875$. The thick line indicates the edge trajectory. **Lower:** Schematic view of phase space. The surface separates initial conditions which relaminarize from those which become turbulent. An edge trajectory visiting three pink states is shown schematically [2].

(energy in the streamwise-dependent part of the flow) as a function of time is shown in figure 7. The edge energy is seen to quickly level off indicating an edge state with *constant* E_{3d} which corresponds to a TW solution (labelled $C3\_1.25$ in [2] and later renamed $N2$ in [10]). This finding was significant in two ways. It was the first identification of a TW using this technique (see also [13] who found a steady state in small geometry plane Couette flow at about the same time) but it was not the expected TW. Calculations in [5] had identified that a lower branch TW (in their nomenclature $2b\_1.25$, now known as $S2$) only had one unstable direction indicating that this would be a relative attractor in the edge. However, the TW found possessed an additional mirror symmetry never seen before. Thus it was the first demonstration of multiple edge states. Secondly, the realisation that such 'highly symmetric' TWs could exist led to whole new families of TWs being quickly discovered thereafter [10]. As Re increases, these waves turn out to appear *before* the original less-symmetric TWs found in [4, 20]. Plausibly, these latter waves arise from the former in symmetry-breaking bifurcations ([10] shows an example of this - S3 bifurcating off N3 - in their figure 8).

Duguet, Willis & Kerswell [2] also noticed that the edge trajectory in the $5\,D$ case at $Re = 2875$ occasionally dipped to low energy values and smoothened locally (see Figure 8 for an example). They realised that the flows at these local energy minima turn out to be very close to other (lower branch) TWs embedded in the edge but these are now saddles there rather than relative attractors. This clearly reinforces the picture of lower branch TWs embedded in the edge. The picture is then of the edge trajectory transiently visiting the neighbourhood of these (saddle) TWs before ultimately reaching an edge state (see Figure 8).

3.3 Larger Geometries

The bisection technique also works in larger geometries where the edge state turns out to be localised. This was first seen in the '$2+\epsilon$' dimensional model of pipe flow [21] and then in fully 3 dimensional pipe flow [8]: see Figure 9. In both these cases, only spatiotemporally chaotic edge states are found. In plane Couette flow, however, spanwise-localised equilibrium and travelling wave solutions were found [15] in short (streamwise) and wide (spanwise) domains. These spanwise-localised solutions were later found to bifurcate off the spanwise-periodic solutions already known [14] suggesting that all strictly periodic solutions could have connected localised versions too (see the talk by Tobias Schneider in this volume). Opening up the flow even further by considerably lengthening the domain leads to an edge state also localised in the streamwise direction, albeit now chaotic [1, 15]. Intriguingly, this edge state resembles a turbulent spot (although the energies are lower) and highlights the large size of domains needed to see streamwise localisation. Figure 10 illustrates this latter point by comparing the small plane Couette flow domain originally used for edge-tracking [13] ($4\pi \times 2 \times 2\pi$ being the streamwise, cross-stream and spanwise dimensions respectively) with the short, wide domain ($4\pi \times 2 \times 8\pi$) and long, wide domain ($64\pi \times 2 \times 16\pi$) of [15]. It is currently an outstanding question as to whether fully localised equilibrium or TW solutions exist in wall-bounded shear flows.

There are still many issues surrounding the edge and efforts have started in very low dimensional models to unravel these [6, 7, 18].

Summary

- Edge tracking is possible if there are two different types of behaviour (which can easily be discerned). The subcritical scenario is ideal as the laminar and turbulent states co-exist. If these two states are locally stable, the edge represents the laminar-turbulent boundary. If they are not, then the edge is a hypersurface which divides initial conditions which experience different initial behaviour only.
- Edge tracking can find exact solutions if they are relative attractors on the edge or even if they are saddle points on the edge (with luck).
- Edge states act as organising centres for transitional flows.
- Edge states vary with Re, geometry of the flow and what symmetries are imposed.
- Edge tracking just needs a DNS code and patience.

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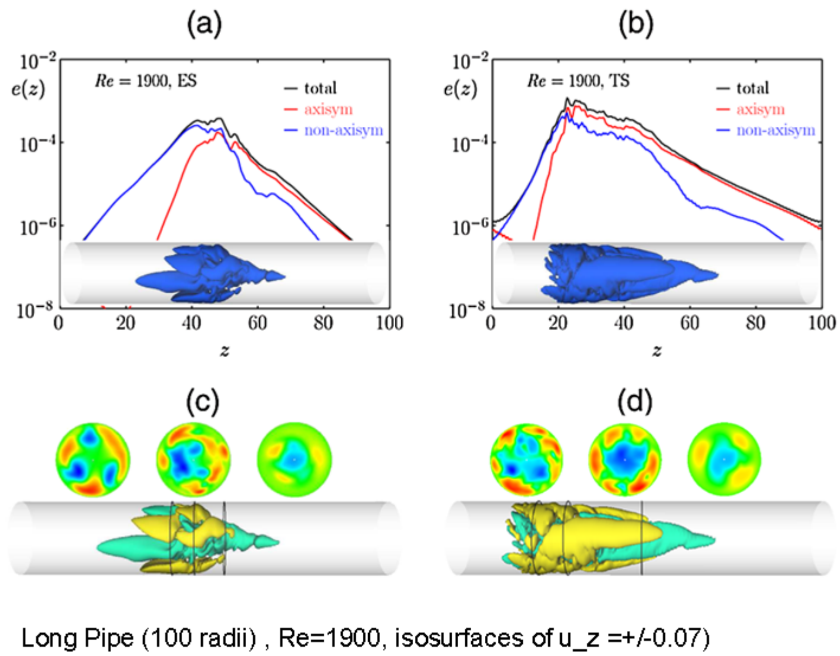

Figure 9: Localized edge state (ES) and turbulent state (TS) at $R_e = 1900$. Energy distribution of (a) the localized edge state and (b) a turbulent state. (c) and (d) show cross-sectional distributions of isosurfaces of the edge state and the turbulent state respectively, from [8].

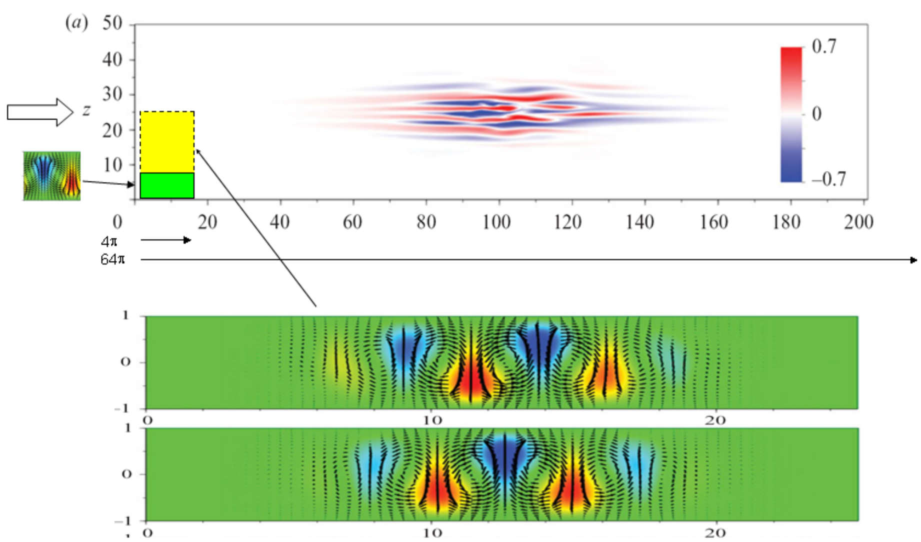

Figure 10: The small domain of [13] in which the edge state is a steady global state is shown as the lower left green rectangle (flow is left to right). The short wide box of [15] is shown as the lower left yellow rectangle (4 times wider than the green rectangle) with both dwarfed by the 16 times longer, wide box which captures a fully localised edge state. Underneath are shown the spanwise localised states found by edge tracking in the short wide box.

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